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AMERICAN
JOURNAL OF MATHEMATICS

FRANK MORLEY, EDITOR

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WITH THE COOPERATION OF

CHARLOTTE A. SCOTT, A. B. COBLE

AND OTHER MATHEMATICIANS

PUBLISHED UNDER THE AUSPICES OF THE JOHNS HOPKINS UNIVERSITY

Πραγμάτων ἔλεγχος οὐ βλεπομένων

VOLUME XLVIII, NUMBER 4

BALTIMORE: THE JOHNS HOPKINS PRESS

B. WESTERMANN & CO., *New York*
E. STEIGER & CO., *New York*
G. E. STECHERT & CO., *New York*
WILLIAM WESLEY & SON, *London*
ARTHUR F. BIRD, *London*
A. HERMANN, *Paris*

OCTOBER, 1926

Entered as second-class matter at the Baltimore, Maryland, Postoffice, acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, Authorized on July 3, 1918

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THE AMERICAN JOURNAL OF MATHEMATICS will appear four times yearly.

The subscription price of the JOURNAL is $6.00 a volume (foreign postage, 25 cents); single numbers, $1.75. A few complete sets of the JOURNAL remain on sale.

It is requested that all editorial communications be addressed to the Editors of the AMERICAN JOURNAL OF MATHEMATICS, and all business or financial communications to The Johns Hopkins Press, Baltimore, Md., U. S. A.

PRINTED BY THE J. H. FURST COMPANY
BALTIMORE, MD.

Regular Maps on an Anchor Ring.

By H. R. Brahana.

Introduction.

The maps obtained by the projection form the center of the faces, edges, and vertices of the regular polyhedra upon a circumscribed sphere are well-known and have many interesting and useful properties. The analogous maps on surfaces of positive genus have received little attention. Professor A. B. Coble and the writer published * a series of maps of five-sided regions with a group of order 120; this list contains all maps of that character. The group of the map, as used in that paper and as it will be used in this, is the group of $(1-1)$ continuous correspondences of the surface, on which the map lies, with itself in which regions correspond to regions, edges to edges, and vertices to vertices.†

Mr. A. Errera ‡ considered the question of regular polyhedra whose surfaces are of positive genus and gave a list of all such polyhedra for surfaces of genus one and genus two, with some examples for surfaces of genus greater than two. In his investigation he retained one of the properties of the regular polyhedra with spherical surfaces, namely, that the faces have the same number of sides and that the same number of faces come together at each vertex. This property is sufficient to characterize the regular polyhedra and regular maps on a sphere and gives a certain measure of regularity to those with surfaces of genus one. From it follows in the one case, but not in the other, the more interesting property that *the group of a regular map of n k-sided regions is of order kn.*§ It is this latter property that we wish to retain and

* "Maps of Twelve Countries with Five Sides with a Group of Order 120 Containing an Ikosahedral Subgroup," *American Journal*, Vol. 48 (1926), pp. 1-20.

† In the paper referred to the group was called the group of displacements (p. 19) of the map into itself. The displacement may not be carried out in two dimensions keeping the correspondence continuous throughout.

‡ *Sur les Polyèdres Réguliers de l'Analysis Situs*, Bruxelles, 1922.

§ That the two properties are not equivalent in the case of an anchor ring may be seen by reference to Mr. Errera's figures 13 and 17. In figure 17 there is a polyhedron with seven hexagonal faces with a group of order 42; in figure 13 is another with seven hexagonal faces with a group of lower order for it contains no rotation about 3 putting 1 into 2.

that we shall use to define "regular map." Obviously, a map regular in this sense will be regular in the other and our maps will be included among Mr. Errera's. The order of the group would seem to be a reasonable measure of the regularity of the map, and according to this criterion our maps will have the greatest possible regularity.

In his discussion of the genus of a group Burnside * considers a problem similar to this. The maps given there, as well as those given by Dyck † to whom Burnside refers, have a certain lesser degree of regularity. These maps are made up of funadmental regions of the groups in question and the order of the group is one-half the number of regions of the map. The groups are transitive on the "white" and "shaded" regions separately but do not interchange white and shaded regions; they permute the vertices only in sets. The groups of our maps will be transitive on regions, lines, and vertices. We shall see that the groups of the regular maps on an anchor ring are included among the groups of genus one given by Dyck and Burnside, and that the regions of the regular maps may be subdivided in an obvious way to give the maps of fundamental regions which they consider.

The condition used by Mr. Errera together with the generalized Euler formula requires that any regular map on an anchor ring have triangular, quadrangular, or hexagonal regions. In none of these cases is there a restriction on the number of regions. If the regions are triangular they appear six at a vertex; if quadrangular, four at a vertex; and if hexagonal, three at a vertex. Since two adjacent vertices are joined by a line and two adjacent regions are separated by a line there is a sort of duality between the maps of triangles and the maps of hexagons. The quadrangular maps are self-dual. So to find all the regular maps on an anchor ring we need only to find those with hexagonal regions and those with quadrangular regions.

In § 1 we assume the existence of the regular maps and investigate the groups; from the theory of abstract groups we are able to state conditions on n, the number of regions necessary for the existence of the map. In § 2 we show that these conditions are sufficient. In § 3 we consider the existence of groups of one of the two types,—a description of the other being already in the literature. In § 4 we establish the uniqueness of the map for a given group, and give a few particularities of the maps and groups in question.

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* *Theory of Groups*, Chapters XVIII and XIX (Second Edition).

† "Gruppentheoretischen Studien," *Math. Ann.*, Vol. 20 (1882).

§ 1. THE GROUP G_{6n} OF A REGULAR MAP OF n HEXAGONS, AND THE GROUP G_{4n} OF A REGULAR MAP OF n QUADRANGLES.

The map of n hexagons on an anchor ring contains n regions, $3n$ lines, and $2n$ vertices. Since the group is of order $6n$ it must put a given region a into each of the n regions in six different ways, and therefore must contain an operation of order six leaving a fixed and permuting the regions on its boundary cyclically. It must also contain an operation of order two interchanging a with a neighboring region b and leaving the line between them, (ab), fixed except that its ends are interchanged. These two operations, s'_2 and s_1 respectively, generate G_{6n}, for the two operations with the powers of s_2' and their transforms are enough to carry a into each region in all possible ways. If we denote by v the end of the line (ab) which is the forward end with reference to the direction of rotation effected by s_2' we observe that $s_2's_1$ leaves v fixed and rotates the map putting each region at v into an adjacent region. Since there are three regions at a vertex, the operation $s_2 = s_2's_1$ must be of order three. For purposes that will appear immediately we note that $s_2 = s_2's_1$ is equivalent to $s_2s_1 = s_2'$ and that the group G_{6n} is also generated by s_2 and s_1. We may now state a condition for the existence of a regular map of n hexagons:

A necessary condition that there exist a regular map of n hexagons is that there exist a group of order $6n$ generated by two operators of orders two and three respectively whose product is of order six.

It is our good fortune to have a detailed study of these groups at hand. G. A. Miller * shows that such a group contains an invariant subgroup H which is generated by one or two operators and which is Abelian. The order of H is one-sixth the order of the group, and the group G/H is cyclic. There will exist a map of n hexagons only if there exists a group G_{6n} whose H described above is of order n. Professor Miller discusses the existence of the groups in terms of the order of H.

This knowledge obtained from the groups may be applied in part to the maps. By considering the map alone we may see that its group contains an Abelian subgroup of order n which is generated by one or two substitutions. For, let a be a region of M surrounded by $(b_1c_1d_1b_2c_2d_2)$ in that order. There

* "On the groups generated by two operators of orders two and three respectively whose product is of order six," *Quarterly Journal*, Vol. 33 (1901), pp. 76-79.

exists an operation T_1, which we will call a translation, one of whose cycles contains b_1ab_2, for any two adjacent regions may be displaced into any two other adjacent regions. The operation performed by T_1 on an adjacent region and consequently on any other region α is similar to that performed on a in that α is transformed into β_1 and β_2 is transformed into α where β_1 and β_2 are opposite countries on the boundary of α. Let T_2 and T_3 be corresponding operations affecting c_1c_2 and d_1d_2. It is evident that T_1 and T_2 are permutable and that $T_1 \cdot T_2$ is equal to T_3 or T_3^{-1}. It follows that the group generated by T_1 and T_2 is Abelian and may be generated by one or two substitutions according as T_2 is or is not a power of T_1. Moreover the group $[T_1, T_2]$ is invariant under the operations of G_{6n} for G_{6n} may be generated by $[T_1, T_2]$ and an operation S_a which rotates the map about a, S_a transforms $[T_1, T_2]$ into itself, and S_a is not in $[T_1, T_2]$. The subgroup $[T_1, T_2]$ is transitive, for it contains operations putting a given region a into each adjoining region, making its order at least n, and it does not contain S_a or any of its powers, making its order at most $6n/6$.

If we turn now to maps of n quadrangles on an anchor ring, we see that matters proceed in a similar way. The group of displacements must contain an element of order four leaving a given region a fixed, and an element of order two leaving the line between a and an adjacent region fixed. Their product leaves one of the vertices of a fixed and so must be of order four if it is not of order two. A consideration of the map shows that the product puts a into an adjoining region and so if of order four. The two operations generate the whole group and we have the theorem:

A necessary condition that there exist a regular map of n quadrangles is that there exist a group of order $4n$ generated by two operators of orders two and four respectively whose product is of order four.

Here we do not have the groups classified as we had in the previous case, but we shall be able to show (§ 2) that any such group contains an invariant subgroup H which is Abelian, generated by one or two operators, and whose order is one-fourth the order of the group. By a consideration of the map we may see that this subgroup H is the group of translations precisely as in the case of maps of hexagonal regions.

§ 2. The Sufficiency of the Conditions of § 1.

We have shown that the existence of a regular map of n hexagons or quadrangles implies the existence of a group of order $6n$ or $4n$ of a certain

type; we proceed to show that the existence of a group of the given type implies the existence of a regular map.

We have a group G_{6n} with an invariant subgroup H of order n. We shall first show that if G_{6n} is represented on n letters the element of order six outside of H which with H generates G_{6n} is of degree $n-1$. This will follow if we prove that this element S_a has $n-1$ conjugates. H is generated by $s_3 = s_1 s_2^2 s_1 s_2$, and $s_4 = s_2^2 s_1 s_2^2 s_1 s_2^2$.* Since H is Abelian any of its elements may be written in the form $s_3^{\alpha} s_4^{\beta}$ where α and β take on the values 0, 1, 2, 3, $\cdots$, $k-1$, k being the order of s_3 and s_4. Moreover, $(s_2 s_1)$ is not in H, for the transform of s_3 by it is s_4^{-1}. If we let s_5 be that substitution of H which satisfies the relation $s_3 s_4 s_5 = 1$, we may verify by carrying out the operations that the transforms of s_3, s_4, s_5 by $(s_2 s_1)$ are respectively s_4^{-1}, s_5^{-1} and s_3^{-1}. From this it follows that $(s_2 s_1) s_3 = s_5^{-1}(s_2 s_1)$, and $(s_2 s_1) s_4 = s_3^{-1}(s_2 s_1)$. Let $(s_2 s_1)$ be denoted by S_a. Then by means of the above relations we see that any transform S_b of S_a by an element of H is

$$S_b = s_3^{-\alpha} s_4^{-\beta} S_a s_3^{\alpha} s_4^{\beta} = s^{-\beta} \cdot s_4^{\alpha-\beta} S_a.$$

To find the number of conjugates of S_a we determine the number of operations of H which transform S_a into itself. This number is the same as the number of distinct solutions of the congruences $-\beta \equiv 0$, $\alpha - \beta \equiv 0$, mod k. The only solution is $\alpha = \beta = 0$, and therefore S_a has $n-1$ distinct conjugates in the group. We will show that H and S_a generate the group by showing that none of the powers of S_a are permutable with every element of H. The transform of S_a^2 by $s_3^{\alpha} s_4^{\beta}$ is $s_3^{-\alpha-\beta} s_4^{\alpha-2\beta} S_a^2$. The number of elements of H permutable with S_a^2 is given by the number of solutions of

$$\begin{cases} -\alpha - \beta \equiv 0 \\ \alpha - 2\beta \equiv 0 \end{cases} \quad \text{mod } k.$$

There are one or three solutions of the system according as k does not or does contain the factor 3. Therefore S_a is of degree $n-1$ or $n-3$. The transform of S_a^3 by $s_3^{\alpha} s_4^{\beta}$ is $s_3^{-2\alpha} s_4^{-2\beta} S_a^3$. S_a^3 is permutable with one or four elements of H according, as k is odd or even, and therefore S_a^3 is of degree $n-1$ or $n-4$. The remaining powers of S_a are inverses of those already considered and so need no further investigation. Hence we see that the

* Cf. Miller: *loc. cit.*

group $[H, S_a]$ is G_{6n} and that if G_{6n} is represented on n letters S_a will be of degree $n-1$. Also, S_a is made up of a number of cycles of six letters and at most one cycle each of two letters and of three letters.

We note further that none of the conjugates of S_a can be the inverse of S_a nor can any contain a cycle of six letters which is the inverse of a cycle of S_a. To see the first suppose $S_b = S_a^{-1}$; then $s_3^{-\beta} s_4^{\alpha-\beta} S_a^2 = 1$, which implies that S_a^2 is in H. For the second statement, it would imply that $S_b S_a = s_3^{-\beta} s_4^{\alpha-\beta} \cdot S_a^2$ omits six letters, but it can not be of degree lower than $n-3$. These facts are the essentials of a proof that the map which we shall proceed to construct is on a two-sided surface.

Now let us construct a hexagon in a plane and surround it by hexagons. Let us name the first hexagon a and the surrounding regions cyclically according to the letters of one of the cycles, say the first, of six of S_a which omits a. Then choose an element of period two outside of H which interchanges a with a letter b which is on one of the regions bounding a. Transform S_a into S_b by this operation and bound b by means of the first cycle of S_b, in the same direction (clockwise or counter-clockwise) as was used in bounding a. This may be continued until we have n regions, one with each of the letters used to represent G_{6n}. Each letter, say b, will appear six times in the first cycle of S_a and its conjugates; thus b will appear on the boundary of six regions. When we obtain a country a remote in the plane from b which has b on its boundary we will signify that the proper side corresponds to the side of b along which b touches a. In this way we obtain a polygon in a plane with its sides paired; the process may be carried out so the polygon has no holes but that is not essential. If we join corresponding sides we obtain a two-dimensional manifold with a map of n hexagons on it. The manifold is two-sided because the polygons are all sensed alike and it is not possible to displace a given polygon into itself in such a way as to reverse its sense. Since there are three regions at a vertex the genus of the manifold is 1. Thus we may state the theorem:

Every group generated by two operators of orders two and three respectively whose product is of order six determines a regular map of n hexagons on an anchor ring, where n is one-sixth the order of the group.

Moreover, if we replace regions by vertices in the above discussion, we have,

Every group generated by two operators of orders two and three respectively whose product is of order six determines a regular map of 2n triangles on an anchor ring, where n is one-sixth the order of the group.

The group G_{4n} is generated by two operators s_1 and s_2 of orders two and four respectively whose product is of order four. Let $s_3 = s_2^2 s_1$ and $s_4 = s_2 s_1 s_2$. The group generated by s_3 and s_4 is invariant, for s_3 and s_4 are transformed into s_3^{-1} and s_4^{-1} respectively by s_1 and into s_4 and s_3^{-1} by s_2. Let the group $[s_3, s_4]$ be denoted by H. H is Abelian, for $s_3 \cdot s_4 = s_2^2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2^3 s_1 = s_4 s_3$. H does not contain s_2, for $s_2^{-1} s_3 s_2 = s_4$. The group $[H, s_2]$ is the whole group G_{4n}, for $s_2^2 s_3 = s_1$ and therefore s_1 and s_2, the generators of G_{4n} are in $[H, s_2]$. The order of H is one-fourth the order of G, for none of the powers of s_2 are in H; this is established when we show that s_2^2 is not permutable with every element of H. Any element of H may be written in the form $s_3^{\alpha} s_4^{\beta}$ where α and β take on the values $0, 1, 2, \cdots, k-1$, k being the order of s_3 and s_4. $s_3^{-\alpha} s_4^{-\beta} s_2^2 s_3^{\alpha} s_4^{\beta} = s_3^{-2\alpha} s_4^{-2\beta} s_2^2$ follows from the relations $s_2 s_3 = s_4^{-1} s_2$ and $s_2 s_4 = s_3 s_2$.

Since G contains a subgroup, s_2 and its powers, of order four which is not invariant and contains no invariant subgroup, it may be represented on n letters. We proceed as before to find the degrees of s_2 and its powers. The transform of s_2 by $s_3^{\alpha} s_4^{\beta}$ is $s_3^{-\alpha+\beta} s_4^{-\alpha-\beta} s_2$, and the number of elements of H permutable with it is given by the number of solutions of the system of congruences

$$\begin{cases} -\alpha + \beta \equiv 0 \\ -\alpha - \beta \equiv 0, \end{cases} \mod k.$$

There are one or two solutions of the system according as k is odd or even. Therefore, s_2 is of degree $n-1$ or $n-2$. The number of elements of H permutable with s_2^2 is given by the number of solutions of $-2\alpha \equiv 0$ and $-2\beta \equiv 0$, mod k, and is one or four according as k is odd or even. Therefore, s_2^2 is of degree $n-1$ or $n-4$. From these facts it follows that if G_{4n} is represented on n letters the operator s_2 which we will call S_a omits one letter if k is odd and two letters if k is even. And since S_a^2 omits one letter in the one case and four in the other, it follows that S_a is made up of cycles of four letters for an odd k and of a number of cycles of four and one cycle of two letters if k is even. Hence, n is either of the form $4h$ or of the form $4h+1$.

When k is odd and n is of the form $4h+1$, S_a has $n-1$ conjugates which are all distinct. No cycle of S_a is conjugate to itself or its inverse, for in the first case $S_b \cdot S_a^3 = s_3^{-\alpha+\beta} s_4^{-\alpha-\beta} S_a^4 = s_3^{-\alpha+\beta} s_4^{-\alpha-\beta}$ and in the other

$S_bS_a = s_3^{-\alpha+\beta}s_4^{-\alpha-\beta}S_a^2$ would be of degree at most $n-4$ which is impossible unless $s_3^{\alpha}s_4^{\beta}$ is identity and then $S_b = S_a$. Therefore among a set of conjugates of a given cycle of S_a no cycle appears twice and no two cycles are inverse, every such set contains $4n$ letters and therefore each letter appears exactly four times. By selecting any cycle to bound a quadrangle a and choosing an element of period two which interchanges a with one of the letters of the chosen cycle we may construct a regular map of n quadrangles in the same manner as we constructed the maps of hexagons. The resulting surface will be two-sided and its genus will be 1.

When k is even we must use methods slightly more refined than in the cases so far considered, due to the fact that $S_b \cdot S_a$ is of degree $n-4$ making it impossible to say that S_b contains no cycle inverse to a cycle of S_a. In fact, there exist groups G_{4n} in which S_a contains a cycle of four letters which is transformed into its own inverse.* However, we shall show that S_a contains not more than one such cycle and that every other cycle of four letters has exactly $n-1$ distinct conjugates none of which is its inverse.

We have seen that S_a omits two letters and that it is made up of a number of cycles of four and one cycle of two letters. There is one element besides identity in H which transforms S_a into itself; this must transform a into the other letter that is omitted by S_a. This element is $s_3^{k/2}s_4^{k/2}$ and is of order two; let us denote it by S_H. If S_H transforms a given cycle of S_a into itself it must contain the second power of that cycle; if it contains more than one such second power $S_H \cdot S_a^2$ is of degree at most $n-8$ which is impossible since $s_3^{-\alpha}s_4^{-\beta}(S \cdot S_a^2)s_3^{\alpha}s_4^{\beta} = s_3^{-2\alpha}s_4^{-2\beta}(S_HS_a^2)$ and therefore $(S_HS_a^2)$ has $n/4-1$ conjugates making its degree $n-4$. Hence S_a contains at most one cycle which is self-conjugate. None of the other cycles of four letters is conjugate to its own inverse. For, the operation S_H' transforming one into the other would be of order two and would therefore be permutable with S_a^2. There are just four such operations in H, -1, $s_3^{k/2}$, $s_4^{k/2}$, and $s_3^{k/2}s_4^{k/2}$. The last of these transform S_a into itself and so cannot transform any of its cycles of order four into the inverse of itself or any other. If $s_3^{k/2}$ transforms such a cycle say $(abcd)$ into its inverse $s_3^{k/2}S_as_3^{k/2} \cdot S_a = s_3^{k/2}s_4^{k/2}S_a^2$ must omit the four letters which appear in S_2^2 as $(ac)(bd)$; therefore this last must be in $s_3^{k/2}s_4^{k/2}$. But this implies that $s_3^{k/2}s_4^{k/2}$ transforms the given

* The group Γ_{64} generated by $(bedm)(fhpn)(gloj)(ci)$ and $(ab)(cd)(en)(fm)(gp)(ho)(ij)(kl)$ contains the element $(bodg)(ejml)(fnph)(ak)$ as a conjugate of the first generator in which $(fnph)$ is the transform of $(fhpn)$.

cycle into itself, and, as we have already seen, it is the only cycle of S_a that is so transformed. We have thus shown that when k is even as well as when k is odd there exists a cycle of S_a which has $n-1$ conjugates distinct from the given one and its inverse. Therefore, we may proceed as before to construct a map which will be regular and which will lie on a two-sided surface of genus one. Hence,

Every group G_{4n} generated by two operators of orders two and four respectively whose product is of order four determines on an anchor ring a regular map of n quadrangles having G_{4n} for its group.

§ 3. THE GROUPS G_{4n}.

The groups G_{6n} were studied by Professor Miller and the existence for suitable values of n was established. We shall first summarize his results and then proceed to the corresponding study of the groups G_{4n}. This study exemplifies very well the usefulness of the regular maps in the investigation of certain types of group.

In a group G_{6n} H, the invariant Abelian subgroup generated by one or two operators, is of order n.

1°. Suppose H is cyclic and $n = p^m$, where m is positive and p is prime. There exists no group when $p \equiv 2$, mod 3; there exists one and only one group for every m when $p \equiv 1$, mod 3; and there exists one group when $p = 3$, m being of necessity equal to 1.

2°. Suppose H is cyclic and $n = 3^{a_0}p_1^{a_1}p_2^{a_2}, \cdots$, where $p_1, p_2, \cdots$ are distinct primes. The p's must be of the form $3h+1$, and there exists at least one group for every such value provided a_0 is 0 or 1.

3°. Suppose H is non-cyclic and $n = p^m$, where p is prime. If $p \equiv 1$, mod 3, there exists a group G_{6n} for every H of type $(m-\gamma, \gamma)$; if $p \equiv 2$, mod 3, there exists a group for every even value of m, H being of type (a, a), $2a = m$; if $p = 3$, there exists a group for every value of m greater than 1, where H is of type $(m'-1, m')$ or of type (m', m').

4°. Suppose H is non-cyclic and $n = 3^{a_0}p_1^{a_1}p_2^{a_2}, \cdots$, the p's being distinct primes. Then there exists a group G_{6n} provided there exist groups of orders 6.3^{a_0}, $6p_1^{a_1}$, $6p_2^{a_2} \cdots$ generated by two operators of orders two and three whose product is of order six.

We shall investigate the groups G_{4n} also according to the character of H.

1′. Suppose H is cyclic and $n = p^m$, where p is prime. The generating operator s_3 of H is transformed into a power of itself by s_2, an operator of order four. The exponent k of this power must therefore satisfy the relation $k^4 \equiv 1$, mod p^m, and since $k \not\equiv 1$, mod p^m it follows that $k^2 \equiv -1$, mod p^m. This congruence will have a solution if and only if $p \equiv 1$, mod 4. Conversely, if $p = 4h + 1$, the operator s_3 may be written as a cycle of n letters, k may be determined and the kth power of s_3 written, and from these two and the relation $s_2^{-1}s_3s_2 = s_3^k$ we may write the operator s_2. From $s_1 = s_2^2 \cdot s_3 = s_2^2 \cdot s_2^2 s_1$ we may obtain s_1, which will be of order two.

2′. Suppose H is cyclic and $n = 2^{a_0}p_1^{a_1}p_2^{a_2} \cdots$, where $p_1, p_2, \cdots$ are distinct primes. Then H is the direct product of subgroups of orders $2^{a_0}p_1^{a_1}$, $p_2^{a_2}$, $\cdots$ and it follows that the group G_{4n} will exist only if $a_0 = 0$ or 1 and each of the p's is of the form $4h + 1$. Conversely, if the p's are of that form G_{4n} will exist, for we may write the subgroups on distinct letters and for each determine the s_1 and the s_2 as above. The products of the s_1's and the s_2's give respectively operators s_1 and s_2 whose product is of order four; the group generated by them contains an H of the required form. The group as thus represented on n letters will not be transitive, but it may be represented transitively on n letters as follows from the argument of § 2.

3′. Suppose H is non-cyclic and $n = p^m$, where p is prime. Then H is generated by two operators s_3 and s_4. The condition that s_3 and s_4 be independent generators is that there be no relation of the form $s_3^{\beta}s_4^{\delta} = 1$. Suppose that such a relation exists, then $s_3^{\beta} = s_4^{-\delta}$. Since $s_2^{-1}s_3^{\beta}s_2 = s_4^{\beta}$, and $s_2^{-1}s_4^{\beta}s_2 = s_3^{-\beta}$ for every value of β, and since $s_3^{-\beta}$ and $s_3^{+\delta}$, being operators of the same order of a cyclic group, satisfy a relation $s_3^{-\beta} = (s_3^{\delta})^k$, it follows that $s_2^{-1}s_4^{\delta}s_2 = s_2^{-1}s_3^{k\delta}s_2 = s_3^{k^2\delta} = s_3^{-\delta}$. Therefore, $s_3^{\delta} \cdot s_3^{k^2\delta} = s_3^{\delta(k^2+1)} = 1$, whence $\delta(k^2 + 1) \equiv 0$, mod p^m. Since δ is not zero, mod p^m, it follows that $k^2 \equiv -1$, mod p^l, $l < m$. We have already seen that this is possible only if $p \equiv 1$, mod 4. Therefore s_3 and s_4 are independent generators of H whenever $p = 2$ or $p \equiv 3$, mod 4.

When $p \equiv 1$, mod 4, we may consider H to be made up of the product of two groups of order p^{α} having a subgroup of order p^{β} in common, $(\beta < \alpha)$. Such a group is also generated by two independent operators and is of type $(m - \gamma, \gamma)$. We may construct two groups of orders $4p^{m-\gamma}$ and $4p^{\gamma}$ on distinct sets of letters respectively, generated by operators of orders two and four,

and establish a $p^{m-\gamma}p^{\gamma}$ isomorphism between them. The resulting group is of order $4n$ and is isomorphic with each of the groups on distinct letters.* Also, the subgroup H is the direct product of the subgroups $H_{p^{m-\gamma}}$ and $H_{p^{\gamma}}$ of the original groups and so is of the proper form. The group may now be represented transitively in n letters.

Suppose p is not of the form $4h+1$. Then s_3 and s_4 are independent generators of H. The order of s_3 and s_4 is p^{α} and the order of H must be an even power of p. We will show the existence of the group G_{4n} when these conditions are satisfied by means of our regular maps. We will show how to construct the maps for $\alpha=1$ and then show how the maps for succeeding values of α may be obtained from them. We take a square and divide it into p^2 squares by drawing lines parallel to its sides. If we make opposite sides of the original square correspond the polygon defines an anchor ring. The group of translations of this map is evidently of order p^2 and is generated by two independent operators of order p. When $p=2$ the center of the polygon is a vertex, and when $p\equiv 3$, mod 4, the center is a region. In either case the rotation R about the center which puts regions into regions is of order four. The group generated by H and R is of order at least $4p^2$ because it transforms a given region into any other region in four ways, and its order is not more than $4p^2$ because it cannot be greater than the product of the orders of H and R. This establishes the existence of G_{4p^2}. If now we divide each region into p^2 squares we obtain a map of p^4 regions whose group of translations is of order p^4, and the rotation R of the preceding map carries regions into regions in the new map. Thus the new map is regular and establishes the existence of G_{4p^4}. Since this process may be carried on indefinitely we have shown the existence of a group of order $4p^m$ for every value of m when $p\equiv 1$, mod 4, and for every even value of m when p is any other prime. When $p\equiv 1$, mod 4, there will be more than one such group whenever $m>3$.

4′. When H is non-cyclic and $n=2^{\alpha_0}p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$, where the p's are distinct primes, we have, as in the case when H is cyclic, H given as the direct product of its subgroups of orders $2^{\alpha_0}p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots$. Hence, if p_i is not of the form $4h+1$, α_i must be even. In case none of the p's are of this form, the method of 3′ gives immediately the regular map and the corresponding G_{4n}. If but one of the p's is of the form $4h+1$, we may construct the map according to 1′ or 3′ depending on whether or not the sub-

* Burnside: *Theory of Groups* (Second Edition), p. 188.

group of that order is cyclic. Then by dividing each region into a number of squares equal to the product of the remaining $p_i^{a_i}$'s obtain the regular map and the corresponding group. The rotation R in this case comes from the original map and is possible on the new map because of the fact that each region was divided so that there were the same number of new regions along one side of an original region as along another. We have only to consider, therefore, the case where at least two of the p's are of the form $4h+1$. In this case the method of 3′ for p of the form $4h+1$ applies. Therefore, we may first construct the regular map corresponding to the product of all the $p_i^{a_i}$'s of the form $4h+1$ and then divide each region into a number of squares equal to the project of the remaining $p_j^{a_j}$'s to obtain a regular map of n regions and so establish the existence of the corresponding group G_{4n}.

§ 4. Uniqueness of the Map for a Given Group. A Group of Order 168.

As a result of our definition of regular map it follows that every such map has a group. From the facts that a transformation of the map into itself is completely determined by the designation of the two regions into which two adjacent regions are transformed, that every pair of adjacent regions is transformed into every other pair of adjacent regions by some operation of the group, and further that the group is generated by two such operations, it follows that the group of a map is unique. Conversely, the method of constructing a map from a given group, described in § 2, obviously leads to a unique map, for the only arbitrariness in the construction was a choice of the sense of the boundary of a and a choice of the cycle of S_a used to bound it. A choice of the opposite sense of the boundary of a would give a map identical with the other when viewed from the other side of the surface, which is permissible since we are dealing only with two-sided surfaces. A different choice of the cycle of S_a would give a map identical with the first except for an interchange of names of regions, for all the cycles of six letters of S_a were alike in the case of the hexagon maps and in the case of the quadrangle maps all the cycles of four letters of S_a were alike except one which could not be used to bound a because it was conjugate to its own inverse. There remains then only the possibility of constructing two maps, at least one by some other method, which though distinct have the same group. There is, of course, a map of triangles having the same group as a map of hexagons, but if the maps have the same kinds of regions this is impossible. For, the generating operators being the same, and there being but one conjugate set of subgroups

of order six in G_{6n}, the maps can differ only by an interchange of regions. Likewise, in G_{4n} there is but one conjugate set of subgroups of order four if n is odd, and there can be but one map of quadrangles with a given group. When n is even there are two conjugate sets of subgroups of order four in G_{4n}, as we have seen in § 2. Every group G_{4n} may be written in the form H, Hs_2, Hs_2^2, Hs_2^3. The inverses of the elements of the co-set Hs_2 all appear in the co-set Hs_2^3, $(s_3^\alpha s_4^\beta s_2)^3 = s_3^\beta s_4^{-\alpha} s_2^3$. When n, and consequently k, the order of s_3 and s_4, is even, the co-set Hs_2 contains two sets of operators one of which is made up of the conjugates of s_2, for, as we have seen, the transform of s_2 by $s_3^\alpha s_4^\beta$ is $s_3^{-\alpha+\beta} s_4^{-\alpha-\beta} s_2$ in which the exponents of s_3 and s_4 are both odd or both even. The operator $(s_1 s_2)$ is in the co-set Hs_2^3, for $s_3^{-\alpha} s_4^{-\beta}(s_1 \cdot s_2) s_3^\alpha s_4^\beta = s_3^{-\alpha-\beta-1} s_4^{\alpha-\beta} \cdot s_2^3$. Thus the co-set Hs_2^3 may also be written $H(s_1 s_2)$. $H(s_1 s_2)$ must also consist of two conjugate sets of operators of order four, being made up of the inverse of operators in Hs_2. The conjugate set which contains $s_1 s_2$ does not contain the inverse of s_2, for $s_3^{-\alpha-\beta-1} s_4^{\alpha-\beta} s_2^3$ can equal s_2^3 only if k is odd contrary to hypothesis. Hence, Hs_2 is made up of elements $s_3^x s_4^y s_2$ and the inverses of $s_3^x s_4^y (s_1 s_2)$ in both of which cases $x + y$ is even. We may now establish a simple isomorphism of the group with itself by letting the operators $s_3^x s_4^y s_2 (x + y \equiv 0, \mod 2)$ correspond to the operators $s_3^y s_4^{-x} s_2 (s_1 s_2)$. Any operator of the group transforms corresponding operators into corresponding operators, for $s_3^{-\alpha} s_4^{-\beta} (s_3^x s_3^y s_2) s_3^\alpha s_4^\beta = s_3^{-\alpha+\beta+x} s_4^{-\alpha-\beta+y} s_2$ and $s_3^{-\alpha} s_4^{-\beta} (s_3^y s_4^{-x} s_1 s_2) s_3^\alpha s_4^\beta = s_3^{-\alpha-\beta+y} s_4^{\alpha-\beta-x} s_1 s_2$. As a result of the foregoing we see that two maps with the same group but with regions representing G_4's of different conjugate sets are identical.

From the foregoing discussion we also see that in the quadrangular maps of an odd number of regions every operation which leaves a region fixed leaves a vertex fixed also, in the maps of an even number of regions the subgroups leaving a region fixed do not leave a vertex fixed. The map which we used in § 3, 3′ to establish the existence of G_{16} is an illustration of this fact, the rotation R about the vertex at the center of the polygon leaving no region unaltered. The rotation R about the central region of the polygon used to establish the existence of the group G_{4p^2} where $p \equiv 3, \mod 4$, leaves fixed the vertex of the map to which the four corners of the polygon correspond.

It is of some interest to note that there are distinct maps of n hexagons (or quadrangles) for particular values of n. For example, a map of 49 hexagons may have its group of translations cyclic or the product of two subgroups of order 7. The existence of two distinct groups of the same order implies the existence of two distinct regular maps of the same number of regions. Another set of maps of interest for curiosity's sake is comprised of

those obtained for small values of n. The map of one hexagon with opposite sides joined lies on an anchor ring and is regular, admitting six rotations into itself. Its group may not be represented on the symbol for one region; this and like exceptions follow from the fact that the subgroup leaving one region fixed is itself invariant, as in this case, or contains an invariant subgroup. The maps of three and four hexagons and the maps of one and four quadrangles require special treatment in the choice of elements in terms of which the groups may be represented.

Another fact of interest from the concrete point of view from which we started is that not all values of n may be used for the number of regions of a regular map. There is no regular map of 8, 10, or 11 hexagons, no map of 14 hexagons although there are maps of 7, 21, and 28 hexagons.

The map of 28 hexagons, M_{28}, is of interest from the fact that all the groups of order 168 have been listed and at least one of them has been studied in a similar connection.* The map M_{28} is obtained from the map of seven hexagons M_7, used by Heffter † to show the existence of a map on the anchor ring which would require seven colors, by inserting a new hexagon on each of the lines of the original map.‡ The group of M_7 is the metacyclic group on seven letters. If the regions of M_7 are designated by 0, 1, $\cdots$, 6 the transformations of M_7 into itself are given by $x' \equiv ax + b$, mod 7, where $a \not\equiv 0$. On M_{28} we may designate a region by two coordinates (x, y). Let the x's of the seven regions that come from M_7 remain unchanged and the y's of these regions be zero. The rotation about the region 0 of M_7 permutes the 21 lines in three sets of six and one set of three, thus associating three regions of M_{28} with (0, 0). Let the x's of these three regions be zero and the y's be 1, 2, and 4 in any order. The transformation $x' \equiv x + 1$ of M_7 defines a transformation on M_{28}. Let this transformation be $\begin{cases} x' \equiv x + 1 \\ y' \equiv y, \end{cases}$ mod 7. The transformation is of period seven and by means of it we may determine the coordinates of the remaining regions from (0, 0), (0, 1), (0, 2), and

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* Miller, "Determination of all the groups of order 168," *Am. Math. Monthly*, Vol. 9 (1902), pp. 1-5. Our group is the only one with seven subgroups of order 8 and 28 subgroups of order 3 (cf. p. 4). Klein, "Ueber die Transformation siebenter Ordnung der Elliptischen Funktionen," *Math. Ann.*, Vol. 14 (1879), pp. 428-471. This group is the simple group of order 168, and the surface is of genus 3.

† "Ueber das Problem der Nachbargebiete," *Math. Ann.*, Vol. 38 (1891), p. 477.

‡ This method of obtaining a map from a given one was used by Franklin, "The Four-Color Problem," *American Journal*, Vol. 44 (1922), p. 235.

(0, 4). The group of M_{28} contains a subgroup of order 42 corresponding to the group of M_7. This subgroup is given by $\begin{cases} x' = ax + b \\ y' = 2y, \end{cases}$ mod 7.

The transformations

$$\begin{cases} x' = x \\ y' = \dfrac{2y+5}{3y+5} \end{cases} \qquad \begin{cases} x' = x \\ y' = \dfrac{2y+3}{5y+5} \end{cases} \qquad \begin{cases} x' = x \\ y' = \dfrac{2y+6}{6y+5}, \end{cases} \quad \text{mod } 7,$$

together with identity form a group of order four which permutes the y's in pairs. The product of this group and G_{42} gives G_{168} of the new map. The group is generated by

$$s_1 \equiv \begin{cases} x' = 6x + 3 \\ y' = \dfrac{2y+5}{3y+5} \end{cases} \quad \text{and} \quad s_2 \equiv \begin{cases} x' = 3x \\ y' = 2y, \end{cases} \quad \text{mod } 7,$$

whose product $s_1 s_2$ is $\begin{cases} x' = 4x + 2 \\ y' = \dfrac{4y+3}{3y+5}, \end{cases}$ mod 7.

The group has an invariant subgroup of order 28 generated by two operators of order 14 having a subgroup of order 7 in common. It contains 28 cyclic subgroups of order 6 which are all conjugate. These 28 subgroups are grouped in seven sets of four all four having the same element of order two. Each of the groups of four has associated with it six more elements of order two and the 23 elements together with identity constitute a subgroup of order 24 which transforms the four regions among themselves. Each G_{24} has an invariant G_8 whose elements are all of order two, and the 7 G_8's have a G_4 in common which is invariant under the whole group. H, the invariant subgroup of order 28, has three cyclic subgroups of order 14 which have G_7 in common. Each of the G_{14}'s contains also one of the operators of G_4 and its remaining elements are of order 14. There are in G_{168}

18 operators of order 14, 6 operators of order 7,
56 operators of order 6, 56 operators of order 3,
31 operators of order 2, and identity.

The map M_7 has been mentioned as being given by Heffter; it is also essentially the map given by Dyck (*loc. cit.*). If the twelve triangles of his map corresponding to a subgroup of order 6 be grouped in a single region we

have the seven hexagons. We have in a way reduced the graphical representation to the simplest form, for G_{42} cannot be represented on fewer than 7 letters; in general, none of the groups with which we have been dealing can be represented on fewer than n letters. It is to be emphasized, however, that the groups which we consider are of a restricted type. The maps given by Dyck can be used to represent graphically any discontinuous group generated by a finite number of operators each of finite order, whereas the groups amenable to our treatment must be generated by two operators of which one is of order two.

Determination of the Type of the Tricrunodal Quartic by Means of Its Invariants.

By L. T. Moore.

The principal object of this paper is the determination of the type of a tricrunodal quartic when its parametric equations are given.

The projective properties of the rational quartic may be derived from the fundamental involutions of the curve. This fundamental involution, to which all line sections are apolar, is a pencil of quartics, $(aT)^4 + \lambda(bT)^4 = 0$. The two-rowed determinants of the matrix $\left\| \begin{matrix} a_0 a_1 a_2 a_3 a_4 \\ b_0 b_1 b_2 b_3 b_4 \end{matrix} \right\|$ of the coefficients of the quartics $(aT)^4$ and $(bT)^4$, may be denoted by

$$p_{ij} = \begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix} .$$

All the invariants of the rational quartic may be expressed in terms of these two-rowed determinants. The complete system of invariants for the rational quartic curve is well known,* and it is the purpose of this paper to express the criteria for the determination of the type of a tricrunodal quartic in terms of the invariants of this system.

Before proceeding to a consideration of this problem, we shall outline a method by which a tricrunodal quartic may be recognized as such. The catalecticant of $(aT)^4 + \lambda(bT)^4 = 0$ is a cubic in λ. The roots of this cubic when substituted in the fundamental involution give catalectic quartics, i. e. quartics which have an apolar quadratic. There are, then three quadratics which multiplied by an arbitrary quadratic are apolar to $(aT)^4 + \lambda(bT)^4 = 0$. Therefore, there are three quadratics giving the parameters of three pairs of points on the curve, which with any arbitrary quadratic form a line section. These three quadratics will give the parameters of the nodes. If the roots of all three of these quadratics are real, the quartic is tricrunodal.

* J. E. Rowe, *Transactions Mathematical Society*, Vol. XII (1911), pp. 295-310.

§ I. *Classification of the Tricrunodal Quartic into Five Types.*

The birational quadratic transformation

$$X = I/X', \quad Y = I/Y', \quad Z = I/Z',$$

transforms the conic

$$(1) \qquad aX^2 + bY^2 + cZ^2 + 2fYZ + 2gXZ + 2hXY = 0$$

into the rational quartic

$$(2) \qquad aY^2Z^2 + bX^2Z^2 + cX^2Y^2 + XYZ(fX + gY + hZ) = 0.*$$

This transformation sends the intersections of the conic with the sides of the triangle of reference into the nodes of the quartic. Thus the points of intersection of (1) with $X = 0$ go into a node at the point (1, 0, 0). If the points of intersection of the conic and the sides are real, then the quartic is tricrunodal with its nodes at the vertices of the triangle.

Using as a basis of classification the way in which (1) cuts the sides of the triangle, we can divide all tricrunodal quartics into five types. If, in passing from one point of intersection of (1) with a side of the triangle to the other intersection with that side, we cross two other sides, or the same side twice, we say that the node of (2) determined by these two intersections is opposite; if the two intersections are adjacent on (1), we say the node is adjacent, the type of a quartic is determined by the number of opposite and adjacent nodes it possesses. There are the following types of tricrunodal quartics.†

Type I. Three adjacent nodes.
Type II. Two adjacent nodes, one opposite node.
Type III. Three opposite nodes.
Type IV. One adjacent, no opposite nodes.
Type V. One opposite, no adjacent nodes.

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* Salmon, *Higher Plane Curves*, 1879, p. 254.

† H. Wieleitner, *Theorie der ebenen algebraischen Kurven höherer Ordnung*, 1905, pp. 146-150.

Type I.

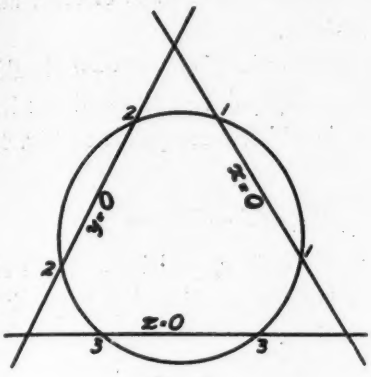

FIG. 1

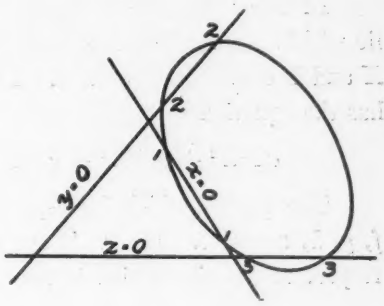

FIG. 2

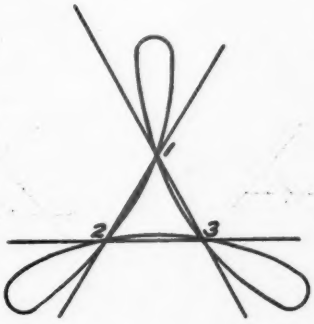

FIG. 3

The points of intersection of the conic and the sides of the triangle may be either all inside the vertices (Fig. 1), or one inside and two outside (Fig. 2). Fig. 3 is a typical curve of Type I. If we call the points of intersection of (1) with $X=0$, 1 1; with $Y=0$, 2 2; with $Z=0$, 3 3, a point describing the conic will pass through them in the order (1 1 2 2 3 3). A point describing the curve of Fig. 3, will pass through the nodes in the same order. The starting point and the direction, of course, have no effect on the number of opposite and adjacent nodes.

The conic through the vertices of the triangle is $fYZ + hXY + gXZ=0$. It cuts $Z=0$ in the points given by $XY=0$. The conic (1) cuts $Z=0$ in the points given by $aX^2 + 2hXY + bY^2 = 0$.

The involution determined by these two quadratics is hyperbolic and must, therefore, have real double points. Their Jacobian is $2(aX^2 - bY^2) = 0$. If the double points of the involution are to be real then a and b must have

the same sign. Since all the involutions on the sides, determined by the conic and the vertices of the triangle, must be hyperbolic, a, b, c, must all have the same sign, which may be taken as positive.

In order that the two points given by $aX^2 + 2hXY + bY^2 = 0$ should lie within the vertices, h must be negative, since any point so situated has its X and Y coordinates positive. Thus the quartic which corresponds to Fig. 1, has the equation

$$aY^2Z^2 + bX^2Z^2 + cX^2Y^2 - 2XYZ(fX + gY + hZ) = 0.$$

The quartic which corresponds to Fig. 2, has a, b, c, positive and one of f, g, h, negative. Fig. 2, is one of three similar figures differing only with respect to the side which is cut inside the vertices.*

Type II.

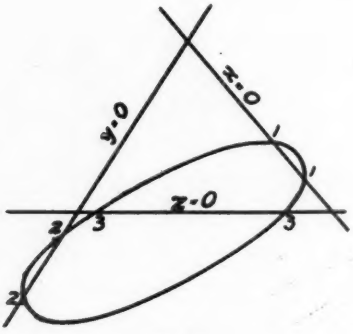

FIG. 4

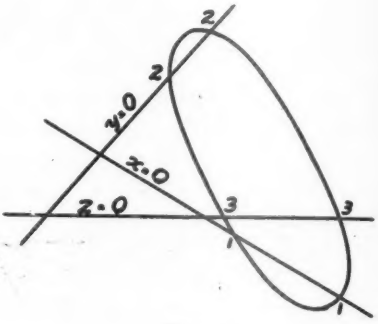

FIG. 5

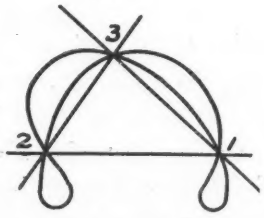

FIG. 6

The order of the points of intersection of the conic and the triangle is (1 1 3 2 2 3). Applying the same arguments as for Type I, we find a, b, c, all positive. For Fig. 4, one of f, g, h, is negative; for Fig. 5, all must be positive. Fig. 6, is a curve of Type II.

* For a full discussion of the properties of the five types of tricrunodal quartics see R. Gentry, *A Dissertation on the Forms of Plane Quartic Curves* (Bryn Mawr), 1896, pp. 17-25.

Type III.

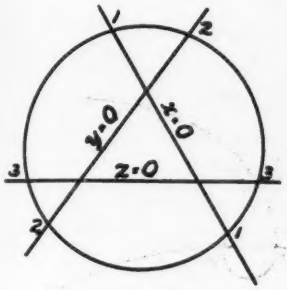

FIG. 7

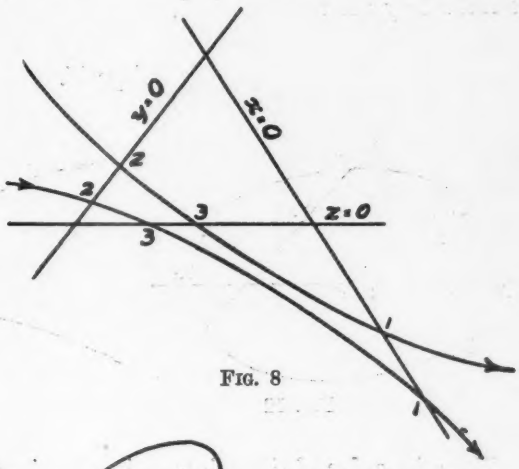

FIG. 8

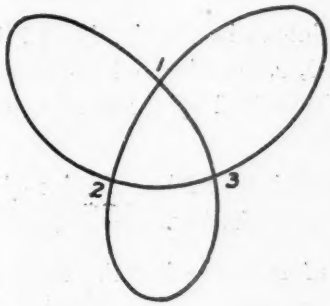

FIG. 9

The order of the points of intersection is (1 2 3 1 2 3); *a, b, c,* are all positive; *f, g, h* are all positive (Fig. 7), or one positive and two negative (Fig. 8).

Type IV.

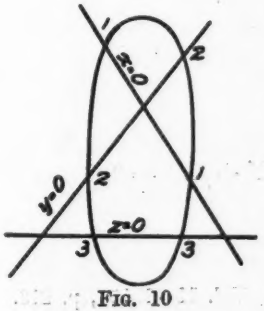

FIG. 10

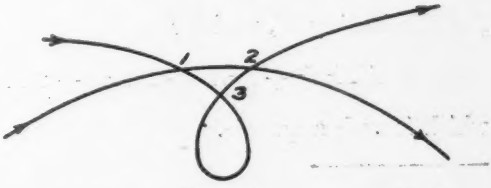

FIG. 11

In this type the order of the points of intersection is (1 3 3 2 1 2); two of a, b, c must be positive and one negative; the signs of f, g, h cannot be determined by the usual method.

Type V.

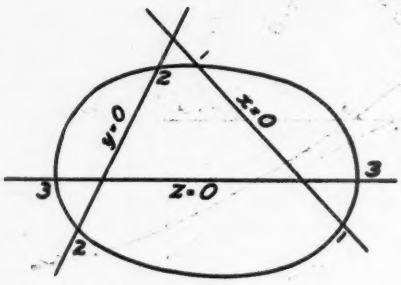

FIG. 12

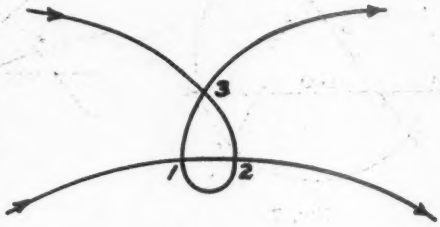

FIG. 13

Here the order of the points of intersection is (1 2 3 2 1 3). This type like Type IV has one of $a, b, c,$ negative and two positive, and, again, the signs of f, g, h cannot be determined by the usual method.

§ II. *The Tricrunodal Quartic as a Plane Section of the Steiner Quartic Surface.*

When the tetrahedron of reference has the three double lines of the surface as edges, and as a fourth plane, the plane which forms with the other three a tetrahedron of unit volume, the equation of the Steiner Quartic Surface may be written as

$$X^2Y^2 + X^2Z^2 + Y^2Z^2 - 2XYZW = 0.*$$

A tricrunodal quartic with nodes at the vertices of the triangle of reference is given by the equations

$$X^2Y^2 + X^2Z^2 + Y^2Z^2 - 2XYZW = 0$$

and

$$aX + bY + cZ + dW = 0.$$

For eliminating W, we have

$$d(X^2Y^2 + X^2Z^2 + Y^2Z^2) + 2XYZ\ (aX + bY + cZ) = 0,$$

which is such a quartic.

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* Salmon-Rogers, *Analytic Geometry of Three Dimensions*, Vol. II (1915), p. 213.

If the equation of the plane is taken in the form

$$K_0X + K_1Y + K_2Z + K_3W = 0$$

and the equation of the Steiner surface in the form

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} + \sqrt{W} = 0,$$ *

(In this form the surface is referred to its tropes—tangent planes that meet the surface in a conic—as a tetrahedron of reference), the symmetric functions of the K's are found to be connected with the invariants of the fundamental involution of the rational quartic curve. Since we are primarily interested in the classification of the tricrunodal quartics by means of these invariants, we will first try to find criteria for distinction in terms of the symmetric functions of the K's, which are invariants of the surface, and then to express these in terms of the invariants of the fundamental involution.

The surface

$$X^2Y^2 + Y^2Z^2 + X^2Z^2 - 2XYZW = 0$$

has four tropes

$$X + Y + Z + W = 0, \quad X - Y + Z - W = 0,$$
$$X - Y - Z + W = 0, \quad X + Y - Z - W = 0.$$

If we refer our surface to these four planes as a tetrahedron of reference, its equation becomes

$$(X^2 + Y^2 + Z^2 + W^2 - 2XY - 2XZ - 2XW - 2YZ - 2YW - 2ZW)^2 - 64XYZW = 0$$

which is the rationalized form of

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} + \sqrt{W} = 0.$$

The plane $aX + bY + cZ + dW = 0$ becomes

$$X(a + b + c + d) + Y(a - b - c + d)$$
$$+ Z(-a - b + c + d) + W(-a + b - c + d) = 0.$$

Therefore

$$K_0 = a + b + c + d, \quad K_1 = a - b - c + d,$$
$$K_2 = d - a - b + c, \quad K_3 = d + b - c - a.$$

The symmetric functions of the K's are

* *Ibid.*, p. 171.

$$S_1 = K_0 + K_1 + K_2 + K_3 = 4d$$
$$S_2 = K_0K_1 + K_0K_2 + K_0K_3 + K_1K_2 + K_1K_3 + K_2K_3 = 2(3d^2 - a^2 - b^2 -$$
$$S_3 = 4[d^3 - d(a^2 + b^2 + c^2) + 2abc]$$
$$S_4 = a^4 + b^4 + c^4 + d^4 - 2a^2b^2 - 2a^2c^2 - 2a^2d^2 - 2b^2c^2 - 2b^2d^2 - 2c^2d^2 + 4abcd.$$

From the equation

$$d(X^2Y^2 + Y^2Z^2 + X^2Z^2) + 2XYZ\,(aX + bY + cZ) = 0$$

the only curves possible are those of Types I, II, and III, since the signs of all the square terms are positive. Considering d to be positive, for these three types a, b, c have the following restrictions as to sign.

Type I			Type II			Type III		
a	b	c	a	b	c	a	b	c
$-$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$-$
$+$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$-$
$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$+$
$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$

The product $abcd$ is positive for Types II and III and negative for Type I. Furthermore,

$$64abcd = S_1^4 - 4S_1^2S_2 + 8S_1S_3.$$

This invariant is, therefore, positive for Types II and III and negative for Type I.

The discriminant of the conic from which the quartic can be derived by the transformation of Part I is

$$D = d(a^2 + b^2 + c^2) - 2abc - d^3.$$

D is positive for Types I and II and negative for Type III.* This is because the sign of D determines the reality of the abnodal tangents of the quartic. Furthermore,

$$S_3 = -4D,$$

hence, for Types I and II S_3 is negative, and for Type III S_3 is positive.

We now have the following criteria which are sufficient to distinguish between curves of Types I, II, and III.

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* R. Gentry, *On the Forms of Plane Quartic Curves*, p. 15.

Type I

$S_3 < 0$

$S_1^4 - 4S_1^2S_2 + 8S_1S_3 < 0$

Type II

$S_3 < 0$

$S_1^4 - 4S_1^2S_2 + 8S_1S_3 > 0$

Type III

$S_3 > 0$

$S_1^4 - 4S_1^2S_2 + 8S_1S_3 > 0$

Curves of Types IV and V are given by the equations

$$X^2Y^2 + X^2Z^2 - Y^2Z^2 + 2XYZW = 0,$$
$$a'X + b'Y + c'Z + d'W = 0.$$

In order to get the surface which gives Types I, II, and III from this surface, we apply the transformation

$$X = X', \quad Y = iY', \quad Z = iZ', \quad W = W'.$$

Thus we have

$$a = a', \quad b = ib', \quad c = ic', \quad d = d'$$

and, therefore,

$$S_1 = 4d'$$
$$S_2 = 2(3d'^2 + b'^2 + c'^2 - a'^2)$$
$$S_3 = 4(2a'b'c' + d'^3 + d'\ (b'^2 + c'^2 - a'^2)) = -4D'.$$

For Type IV, $D' > 0$, and for Type V, $D' < 0$. * Therefore for Type IV, $S_3 < 0$. Type V, $S_3 > 0$.

We now have sufficient criteria to determine the type of a tricrunodal quartic, provided we know whether it is of Types I, II, III, of Types IV, V.

We have between the planes $aX + bY + cZ + dW = 0$, which cuts out curves of Types I, II, III, and $a'X + b'Y + c'Z + d'W = 0$, the relations

$$a = a', \quad b = ib', \quad c = ic', \quad d = d'.$$

Consider the invariants

(1) $8S_2 - 3S_1^2 = -16(a^2 + b^2 + c^2) = -16(a'^2 - b'^2 - c'^2)$

(2) $S_4 - \frac{1}{4}(S_2 - \frac{S_1^2}{4})^2 - 1/16\ (S_1^4 - 4S_1^2S_2 + 8S_1S_3)$
$= -4(b^2c^2 + a^2b^2 + a^2c^2) =$
$-4(b'^2c'^2 - a'^2b'^2 - a'^2c'^2).$

For a curve of Types I, II, III (1) and (2) must be negative. For a curve of Types IV, V, (1) is positive unless $a'^2 > b'^2 + c'^2$. (2) is positive

* *Ibid.*, p. 17.

if a' is greater than b' or c'. Therefore for Types IV, V, one or both of (1) and (2) must be positive.

The criteria distinguishing the five types are now complete and may be stated as follows:

Type I.

$$\mathfrak{I}' \equiv S_1^4 - 4S_1^2S_2 + 8S_1S_3 < 0$$
$$\mathfrak{I}'' \equiv 8S_2 - 3S_1^2 < 0$$
$$\mathfrak{I}''' \equiv 16S_4 - 4\left(S_2 - \frac{S_1^2}{4}\right)^2 - (S_1^4 - 4S_1^2S_2 + 8S_1S_3) < 0$$
$$S_3 < 0.$$

Type II.

$$\mathfrak{I}' > 0 \qquad \mathfrak{I}''' < 0$$
$$\mathfrak{I}'' < 0 \qquad S_3 < 0$$

Type III.

$$\mathfrak{I}' > 0 \qquad \mathfrak{I}''' < 0$$
$$\mathfrak{I}'' < 0 \qquad S_3 > 0$$

Type IV.

$$\mathfrak{I}'' \text{ or } \mathfrak{I}''' > 0 \qquad S_3 < 0$$

Type V.

$$\mathfrak{I}'' \text{ or } \mathfrak{I}''' > 0 \qquad S_3 > 0$$

§ III. *The Connection Between the Invariants of the Steiner Quartic Surface and the Invariants of the Rational Quartic Curve.*

The following table showing the correspondence between the invariants of the Steiner Surface and those of the fundamental involution, is taken from the unpublished notes of Professor Frank Morley.

Singularity	*Invariant of F. I.*	*Invariant of Surface* $\sqrt{X}+\sqrt{Y}+\sqrt{Z}+\sqrt{W}=0$
Triple Point	I_2	S_1S_3
Discriminant of q	I'_2	$S_1S_3 - 16S_4$
Discriminant of $N=0$	$I'_2 - 4I_2$	S_4
Discriminant of $g_2=0$	$I'_2 + 12I_2$	$S_1S_3 - 4S_4$
Undulation	I_4	$U_8 = S_2S_3^2 - S_1S_3S_4 + S_4^2$
Discriminant of $F=0$	$I_4' \equiv I_4 - (I_2' - 36I_2)^2$	$36S_3(S_1^2S_3 - 4S_2S_3 - 8S_1S_4)$
Cusp	I_6	$S_3^2(S_1^2S_4 - S_1S_2S_3 + S_3^2)$
Tacnode	$I_6' \equiv I_2I_4 - I_2(I_2' - 4I_2)^2 - 64I_6$	S_3^4

Let us write

$$4I_2 = S_1S_3M$$

and

$$I_2' = (S_1S_3 - 16S_4)M$$

where M is some constant. These two equations account for the first four relations in the table. Then

$$I_2' - 36I_2 = -8(S_1S_3 + 2S_4) \cdot M$$

If we put

$$I_4 = U_8M' \qquad \text{then}$$

$$I_4' = I_4 - 64(S_1S_3 + 2S_4)^2M^2.$$

I_4' must be free from S_4^2; but in order for

$$I_4' = U_8M' - 64(S_1S_3 + 2S_4)^2M^2$$

to be free of S_4^2, $M' = (16M)^2$.

If $\quad I_6 = (S_1^2S_4 - S_1S_2S_3 + S_3^2)S_3^2M''$

then $\quad I_6' = \frac{1}{4}S_1S_3M\{(16M)^2(S_2S_3^2 - S_1S_3S_4 + S_4^2) - (16M)^2S_4^2\}$

$$- 64(S_1^2S_4 - S_1S_2S_3 + S_3^2)S_3^2M''$$

should be a multiple of S_3^4. This is true if $M'' = -M^3$.

The four equations connecting the invariants of the surface and those of the fundamental involution are, therefore:

$$4I_2 = MS_1S_3$$

$$I'_2 = M(S_1S_3 - 16S_4)$$

$$I_4 = (16M)^2(S_2S_3^2 - S_1S_3S_4 + S_4^2)$$

$$I_6 = M^3S_3^2(S_1S_2S_3 - S_1^2S_4 - S_3^2).$$

These equations may be solved for S_1^4, S_2^2, S_4, $S_1^2S_2$, S_1S_3. The results are as follows:

$$16MS_4 = 4I_2 - I_2'$$

$$MS_1S_3 = 4I_2$$

$$MS_1^4 = \frac{4(16)^3 I_2^4}{I_6'}$$

$$MS_1^2S_2 = \frac{4I_2^2\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2\}}{I_6'}$$

$$MS_2^2 = \frac{\{I_4 + 64I_2(4I_2 - I_2') - (4I_2 - I_2')^2\}^2}{1024I_6'}$$

It is now possible to express the invariants of Section II, in terms of I_1, I_2', I_4, I_6, provided the degree in K is raised to four. This may be accomplished by multiplying by the required power of S_1, as this may always be regarded as positive. The results are as follows:

$$MS_1S_3 = 4I_2$$

$$\mathfrak{I}' \equiv M(S_1^4 - 4S_1^2S_2 + 8S_1S_3) = \frac{16I_2(I_2I_4' + 2048I_2^3 - 128I_6)}{I_6'}$$

$$\mathfrak{I}'' \equiv M(8S_1^2S_2 - 3S_1^4) = \frac{32I_2^2(I_4' - 128I_2I_2')}{I_6'}$$

$$\mathfrak{I}''' \equiv M\{(16S_4 - 4(S_2 - \frac{S_1^2}{4})^2 - (S_1^4 - 4S_1^2S_2 + 8S_1S_3)\}$$

$$= \frac{1{,}960{,}960I_2^4 + 3154I_2^2(I_4' - 128I_2I_2') - (I_4' - 128I_2I_2')^2 - 256I_6'(I_2' + 28I_2)}{256I_6'}$$

The sign of M, determined by taking a curve known to be of Type II and calculating its invariants, was found to be positive.

Finally, we may tabulate our results as follows:

Type I.	$I_2 < 0$,	$\mathfrak{I}' < 0$,	$\mathfrak{I}'' < 0$,	$\mathfrak{I}''' < 0$,
Type II.	$I_2 < 0$,	$\mathfrak{I}' > 0$,	$\mathfrak{I}'' < 0$,	$\mathfrak{I}''' < 0$,
Type III.	$I_2 > 0$,	$\mathfrak{I}' > 0$,	$\mathfrak{I}'' < 0$,	$\mathfrak{I}''' < 0$,
Type IV.	$I_2 < 0$,	$\mathfrak{I}''$ or	$\mathfrak{I}''' > 0$,	
Type V.	$I_2 > 0$,	$\mathfrak{I}''$ or	$\mathfrak{I}''' > 0$.	

Subgroups of Index p^2 Contained in a Group of Order p^m.

By G. A. Miller.

A necessary and sufficient condition that a group G of order p^m, p being any prime number, contains an invariant subgroup of index p^α which gives rise to an abelian quotient group of type $(1, 1, 1, \cdots)$ is that the index of the ϕ subgroup of G is at least equal to p^α, and when this condition is satisfied the number of such invariant subgroups is equal to the number of the subgroups of order p^α found in the ϕ quotient group of G. The ϕ subgroup of G may therefore be defined as the cross-cut of all the invariant subgroups of G which give rise to abelian quotient groups of type $(1, 1, 1, \cdots)$, and it is determined by the fact that it is the cross-cut of all the invariant subgroups of a given order which give rise to such a quotient group. In the special case when $\alpha = 1$ there is always at least one such invariant subgroup in G, and hence the ϕ subgroup can always be defined with respect to the subgroups of index p. Moreover, whenever G is non-cyclic it contains at least one invariant subgroup which corresponds to a non-cyclic quotient group of order p^2, and hence the ϕ subgroup of every non-cyclic group of order p^m is the cross-cut of all its invariant subgroups of index p^2 which corresponds separately to a non-cyclic quotient group.

Since every pair of invariant subgroups of index p contained in G has an invariant subgroup of index p^2 for its cross-cut, and this cross-cut corresponds to a non-cyclic quotient group of G, it results that there is a $(1, 1)$ correspondence between the invariant subgroups of index p^2 which give rise to non-cyclic quotient groups of G and the sets of $p + 1$ subgroups of G which have separately a common subgroup of index p^2. As the number of such sets of subgroups is equal to the number of subgroups of order p^2 in the ϕ quotient group of G, and this quotient group is abelian and of the type $(1, 1, 1, \cdots)$, it results that, when G has $\beta > 1$ independent generators, this number is either

$$(1 + p + p^2 + \cdots + p^{\beta-1})(1 + p^2 + p^4 + \cdots + p^{\beta-3})$$

or

$$(1 + p + p^2 + \cdots + p^{\beta-2})(1 + p^2 + p^4 + \cdots + p^{\beta-2})$$

as β is odd or even. In particular, this number is always of the form $1 + p + kp^2$, where k is a positive integer or 0. From the fact that the ϕ

subgroup of G is found in each of its subgroups of index p it follows that every subgroup of index p contained in G involves the same number of invariant subgroups which separately give rise to an abelian quotient group of a given order and of the type $(1, 1, 1, \cdots)$. In particular, the number of such invariant subgroup of index p^2 is $1+p+p^2+\cdots+p^{\beta-2}$ whenever $\beta > 1$. As each of them appears in exactly $p+1$ subgroups of index p it results that a necessary and sufficient condition that the number of these subgroups of index p is equal to the number of invariant subgroups of index p^2 which correspond to non-cyclic quotient groups is that $\beta = 3$.

An invariant subgroup of index p^2 which corresponds to a cyclic quotient group appears in only one subgroup of index p under G. A necessary and sufficient condition that G involves such an invariant subgroup is that the order of its commutator subgroup is less than that of its ϕ subgroup, and a necessary and sufficient condition that a subgroup of index p under G contains such an invariant subgroup of G is that the subgroup K generated by the commutator subgroup of G and the ϕ subgroup of this subgroup of index p is of index p under the ϕ subgroup of G. The subgroup K must obviously be either identical with the ϕ subgroup of G or be of index p under this subgroup. In the former case the given subgroup of index p under G involves only invariant subgroups of index p^2 under G which correspond to non-cyclic quotient groups. In particular, it results that *whenever G involves a subgroup of index p which includes an invariant subgroup of index p^2 corresponding to a cyclic quotient group of G it must include exactly $p^{\beta-1}$ such subgroups of index p^2.* Since none of these $p^{\beta-1}$ subgroups appears in any other subgroup of index p under G the number of the invariant subgroups which correspond to cyclic quotient groups of order p^2 must always be of the form $kp^{\beta-1}$, where k is a positive integer or 0, and β is the number of the independent generators of G.

It has now been proved that the invariant subgroups of index p^2 which correspond to cycle quotient groups have in common with such invariant subgroups corresponding to non-cyclic quotient groups the property that the number is constant for all the subgroups of index p in which such subgroups appear. On the other hand, they differ as regards the property that every one of the former appears in only one subgroup of index p while every one of the latter appears in $p+1$ such subgroups. Moreover, G may contain subgroups of index p which do not involve any of the former subgroups while some of its subgroups of this index involve separately $p^{\beta-1}$ such subgroups. On the other hand, if one subgroup of index p involves an invariant subgroup corresponding to a non-cyclic quotient group then every subgroup of

this index must have this property. It may be noted here that the number of the invariants of the commutator quotient group of any prime power group is equal to the number of the invariants of its ϕ quotient group, and hence the latter is simply isomorphic with the group generated by the operators of order p contained in the former.

Some of the results noted above can readily be generalized as follows: If an invariant subgroup of index p^α corresponds to a cyclic quotient group of G then it appears in one and only one subgroup of G whose index is less than p^α. The subgroup of index $p^{\alpha-1}$ in which this invariant subgroup appears contains exactly $kp^{\beta-1}$ invariant subgroups of index p^α which correspond separately to a cyclic quotient group. A proof of these somewhat more general statements follows almost directly from the fact that the cross-cut of all the invariant subgroups of G which correspond to cyclic quotient groups of any order involves the commutator subgroup of G, and all the quotient groups of G corresponding to various such cross-cuts involve the same number of independent generators as G irrespective of the order of this cross-cut. In fact, all the quotient groups of any abelian group with respect to the various subgroups composed of powers of the operators of this abelian group have the same number of independent generators and the ϕ subgroup of an abelian group of order p^m is its quotient group with respect to the subgroup formed by the pth powers of its operators. The number of the subgroups of index p which contain an invariant subgroup of G corresponding to a cyclic quotient group of order p^2 is either zero or $1 + p + \cdots + p^{\gamma-1}$, where γ is the number of the invariants of the commutator quotient group of G which exceed p, since this number is equal to the number of the subgroups of index p in the group generated by the pth powers of the operators in the quotient group of G with respect to the cross-cut of its invariant subgroups giving rise to cyclic quotient groups of index p^2. Hence the following theorem: *The number of the invariant subgroups of a group of order p^m which correspond to cyclic quotient groups of order p^2 is $p^{\beta-1}(1 + p + p^2 + \cdots + p^{\gamma-1})$, where β and γ represent respectively the number of the independent generators of the group and the number of invariants which exceed p of its commutator quotient group.*

Every non-variant subgroup of index p^2 contained in G belongs to a set of p conjugates under G, and every pair of subgroups of such a set generates a subgroup of index p under G which includes the entire set. The cross-cut of such a pair of conjugates is invariant under this subgroup of index p but it is not necessarily invariant under G. There must, however, be at least one set of p conjugate subgroups of index p^2 in G which have a common cross-cut, whenever G contains a non-invariant subgroup of this index. This

cross-cut is therefore invariant under G and of index p^3. To prove this theorem it may be noted that the subgroup H of index p under G, generated by a set of p conjugate subgroups of index p^2, involves a characteristic subgroup formed by the cross-cut of all its subgroups of order p^{m-2}, and that the corresponding quotient group I is abelian and of type $(1, 1, 1, \cdots)$.

Let S represent any operator of G which is not found in H and note the transformations of the co-sets of H under S with respect to the given characteristic subgroup. It will be convenient to speak of these transformations as transformations of the operators of I. Hence we may say that S is not commutative with all the operators of I since it does not transform into itself every subgroup of index p under H. The commutator subgroup arising from S and operators of I cannot have a smaller index than p under I since I and S generate a prime power group. Hence I contains an invariant subgroup under S such that the index of this subgroup under I is p^2 and that it involves a subgroup of index p of the commutator subgroup arising from S and operators of I but does not involve this commutator subgroup. The $p+1$ subgroups of index p under I which have for their common cross-cut this invariant subgroup include one which is invariant under S while the remaining p are conjugate under S. This completes a proof of the following theorem: *If a group of order p^m contains non-invariant subgroups of index p^2 then every set of p such conjugate subgroups generates a subgroup of index p which involves at least one set of p conjugate subgroups of index p^2 whose common cross-cut is of order p^{m-3} and is invariant under the entire group.*

Every non-invariant subgroup of index p^2 appears in one and only one subgroup of index p under G. If there are p and only p such subgroups in G they must have a common cross-cut of order p^{m-3} which is invariant under G according to the theorem noted at the close of the preceding paragraph. The quotient group with respect to this cross-cut is the non-abelian group of order p^3 which involves operators of order p^2, when $p > 2$. When $p = 2$ there is no group of order p^m which contains exactly p non-invariant subgroups of index p^2. Since the p^ath power of every operator in G is found in every subgroup of index p^a it follows that the common cross-cut of all the subgroups of index p^2 found in G involves the p^2 power of all the operators of G. Every non-abelian group of order p^m, $p > 2$, which involves operators of order p^{m-1} has the property that it contains one and only one set of p conjugate subgroups. All its other subgroups are invariant. Hence when $m = 3$ this illustrates the case when G contains one and only one set of p conjugate subgroups of index p^2.

UNIVERSITY OF ILLINOIS.

Mapping by Means of Linear Systems of Curves Invariant under Cremona Involutions.

By Theodore Lake Bennett.

I. Introduction.

1. *Statement of the Problem.* It is known that any Cremona involution in the plane can be reduced,* by a proper Cremona transformation, to either the Geisser involution, the Bertini involution, the harmonic homology, or the Jonquiere involution; in this paper we restrict ourselves to these types. It is known that pairs of corresponding points in an involution can be mapped rationally upon the points of a rational surface; † we determine the most general mapping systems by means of invariant curves. To do this we write a linear system of invariant curves,

$$a_0C_0(x) + a_1C_1(x) + \cdots + a_rC_r(x) = 0, \tag{1}$$

and map the x plane upon a surface z by means of the equations,

$$z_i = C_i(x), \quad i = 0, 1, \cdots, r. \tag{2}$$

In all cases it is convenient to map first upon either a plane or quadric y, and then obtain the general z map as a map of the surface y. In order to write equation (1) we show how to write the general invariant curve, except for a possible extraneous factor which does not affect the mapping. The mappings upon the surfaces y have been considered before in the Geiser and Bertini ‡ cases.

In each involution there is a locus of fixed points J which maps into a locus J'. Theorem I of § 2 leads us to consider systems of contact curves (and surfaces) of J'. It is with these contact systems that this paper is chiefly concerned. In § 7 we connect the contact systems with the perspective curves of Stahl,§ and prove a theorem relating to Aronhold sets.

* Bertini, *Annali di Matematica*, (2), Vol. 8 (1877), p. 244.
† Castelnuovo, *Rendiconti dei Lincei*, (5), Vol. 2 (1893), p. 205.
‡ Wiman, *Mathematische Annalen*, Vol. 48 (1897), p. 195.
§ *Mathematische Annalen*, Vol. 38 (1891), p. 561.

II. THE GEISER INVOLUTION, G.

2. *Properties of G, and a Mapping System.* Select any seven points p_i in the x plane, and let the net of cubics on them be

$$a_0C_0 + a_1C_1 + a_2C_2 = 0. \tag{3}$$

All cubics of the net which pass through an arbitrary point P pass also through another point Q; P and Q are copoints in G. The point p_i is a fundamental point, the directions about p_i corresponding to the points of the cubic of the net having a double point at p_i. Every cubic of the net is an invariant curve. The locus of fixed points, J, is a sextic with a double point at p_i, the tangents at p_i being the same as those of the fundamental cubic corresponding to p_i. G is of order eight, i. e. as P describes a line, Q describes an octavic with a triple point at p_i.

The equations,

$$y_0 = C_0, \quad y_1 = C_1, \quad y_2 = C_2, \tag{4}$$

map pairs of copoints in the x plane upon points of the y plane. To any curve in y corresponds an invariant curve in x, and conversely; therefore, any invariant curve in x can be expressed $(ay)^n = 0$. To cubics of the net correspond lines in y; to the fixed curve J corresponds a non-singular quartic J', since an arbitrary cubic of the net cuts J in four points outside of p_i. The equation of J', when y_i is replaced by C_i, gives rise to a 12-ic in x, which is the square of J.

Since the general invariant curve may be written $(ay)^n = (aC)^n = 0$ we see that the general mapping of the x plane obtained from (2) is equivalent to a mapping from x to y, followed by a rational mapping of the y plane.

Let Γ_1 be a curve in x of order n, with multiplicity r_i at p_i; then Γ_2, the transform of Γ_1 in G, is of order $8n - 3\Sigma r_i$. The curve $\Gamma = \Gamma_1 \cdot \Gamma_2$ is invariant, and maps into Γ', of order $3n - \Sigma r_i$. But Γ has $6n - 2\Sigma r_i$ double points on J, and does not meet J elsewhere (except at p_i); hence Γ' cuts J' at only $6n - 2\Sigma r_i$ distinct points, and therefore touches J' this many times, since Γ' and J' have $4(3n - \Sigma r_i) = 2(6n - 2\Sigma r_i)$ intersections. We have thus proved

THEOREM I. *If an invariant curve Γ is the product of two curves, neither of which is itself invariant, then Γ', the map of Γ, is a contact curve of J'.*

The converse of this theorem is not true, i. e. if K' is a contact curve of J', then the invariant curve K which maps into K' need not be composite; but K must have two coincident points in common with J corresponding to each point at which K' touches J'. Moreover, it is readily seen that such coincident intersections can not arise from tangencies, but K must have a double point on J corresponding to each point of contact of K' and J'.

3. *The Double Tangents of J'.* From Theorem I we see that all composite cubics of the net (3) map into double tangents of J'. If we denote by the symbol $f_n(1^{e_1}2^{e_2}\cdots 7^{e_7})$ any curve of order n with multiplicity e_i at p_i, the 21 composite cubics of the net are of the form $f_1(i_1 i_2)\cdot f_2(k_1k_2k_3k_4k_5)$, where $i_1, i_2, k_1, \cdots, k_5$ are the numbers $1, 2, \cdots, 7$ in some order. The corresponding 21 double tangents of J' we denote by $D_{i_1 i_2}$. The 7 fundamental cubics also map into double tangents of J', since each touches J twice at the corresponding base point; we denote these 7 double tangents by D_{ii}, so that the 28 double tangents are denoted by D_{ik}, $i, k = 1, 2, \cdots, 8$, $i \neq k$.

This mapping gives us not only the complete set of 28 double tangents of J', but it isolates 7 of them—those corresponding to the 7 fundamental cubics. We shall examine the nature of this isolated set.

Let us map the x plane upon a z plane by a quadratic transformation T: $x_i = (a_i z)^2$, $i = 0, 1, 2$. Let the fundamental points in the x and z planes be $\xi_1\xi_2\xi_3$ and $\zeta_1\zeta_2\zeta_3$, respectively (the directions about ζ_i corresponding to the points of $\xi_j\xi_k$), and let the points ξ_i be distinct from p_i, so that p_i is transformed into a unique point q_i. The net of cubics on p_i is thus transformed into a net of sextics on q_i, with triple points at ζ_i. If P in x is transformed into R in z, then in mapping the z plane upon y by $x_i = (a_i z)^2$, $y_i = C_i(x)$ the point R is mapped into P', the same point that arises from P in mapping from x to y by $y_i = C_i(x)$. Therefore, if the map of J in z is $\mathscr{J}$, the map of $\mathscr{J}$ in y is J', and the 7 double tangents of J' which are isolated in the mapping from z to y are the same as the 7 isolated in mapping from x to y, since the directions about q_i correspond to those about p_i. But in the mapping from z to y we have isolated also the loci in y corresponding to the points ζ_i, i. e. the loci in y corresponding to the points of the lines $\xi_j\xi_k$. In order to avoid the introduction of such extraneous loci we take the points ξ_i at 3 of the points p_i, so that the net of cubics on p_i is transformed into a net of cubics on 7 corresponding points. Let T be so chosen that $\xi_i = p_i$, $\zeta_i = q_i$, $i = 1, 2, 3$, and as before let T carry $p_4 \cdots p_7$ into $q_4 \cdots q_7$. Then $\mathscr{J}$ is a sextic with double points at $q_1 \cdots q_7$, and in the mapping from z to y we isolate on J' the 7 double tangents corresponding to the directions

about $q_1 \cdots q_7$, i. e. those which in the x to y mapping we denoted by $D_{23}D_{31}D_{12}D_{48}D_{58}D_{68}D_{78}$. By considering all projectively distinct Cremona transformations with base points at p_i we find all sets of 7 double tangents which can be isolated in this way; any one such set of 7 is called an Aronhold set. Since there are 288 projectively distinct transformations, there are 288 Aronhold sets, which may be written

$$D_{ik_1}D_{ik_2} \cdot \cdots \cdot D_{ik_7}$$
$$D_{i_2i_3}D_{i_3i_1}D_{i_1i_2}D_{jk_1}D_{jk_2}D_{jk_3}D_{jk_4}$$

Of these types we have 8 and 280 sets, respectively. In § 4 will be stated a geometrical property of Aronhold sets.

4. *Contact Conics of J'.* Contact conics of J' must correspond to invariant sextics in x which have double points at p_i and 4 other double points on J. Any such sextic must be of one of the types,

(5)
a) $f_1(i_1) \cdot f_5(i_1k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2)$,
b) $f_2(i_1i_2i_3i_4) \cdot f_4(i_1i_2i_3i_4k_1^2k_2^2k_3^2)$,
c) $f_3(i_1^2k_1k_2k_3k_4k_5) \cdot f_3(i_2^2k_1k_2k_3k_4k_5)$.

A curve of type $f_1(i_1)$, $f_2(i_1i_2i_3i_4)$, or $f_3(i_1^2k_1k_2k_3k_4k_5)$ is any one of a pencil, and when we multiply it by its transform under G we obtain a quadratic system of ∞^1 invariant sextics, giving rise to a quadratic system of ∞^1 contact conics of J'. Since there are 7, 35, 21 systems of sextics of type a), b), c), respectively, J' has $7 + 35 + 21 = 63$ systems of contact conics. Since all composite invariant sextics are included in this list, we obtain in this way all the contact conics of J'.

In each quadratic system of contact conics we expect to find 6 line pairs, i. e. pairs of double tangents. When the line giving rise to type a) passes through p_{i_2} as well as p_{i_1}, the corresponding invariant sextic is $f_1(i_1i_2) \cdot f_2(k_1k_2k_3k_4k_5) \cdot f_3(i_1i_2^2k_1k_2k_3k_4k_5)$ which maps into the line pair $D_{i_1i_2}D_{i_2s}$. Consequently, in the corresponding system of contact conics we find the 6 line pairs

$$D_{i_1i_2}D_{i_2s},\ D_{i_1i_3}D_{i_3s},\ \cdots,\ D_{i_1i_7}D_{i_7s}.$$

From the standard theory of the quartic we know that any two of these line pairs, being two conics of the same contact system, touch J' in 8 points which lie on a conic; the set is called a Steiner complex of double tangents.

By treating types b) and c) similarly, we find that the 63 Steiner complexes may be written

$$D_{i_1k_1}D_{i_2k_1},\ D_{i_1k_2}D_{i_2k_2},\ \cdots,\ D_{i_1k_6}D_{i_2k_6}$$

$$D_{i_1i_2}D_{i_3i_4},\quad D_{i_1i_3}D_{i_2i_4},\quad D_{i_1i_4}D_{i_2i_3},\quad D_{k_1k_2}D_{k_3k_4},\quad D_{k_1k_3}D_{k_2k_4},\quad D_{k_1k_4}D_{k_2k_3}.$$

Of these types there are 28 and 35 complexes respectively. Evidently, any pair of double tangents lies in one and only one Steiner complex.

In the theory of the quartic, if three double tangents have their contacts on a conic, they are said to form a syzygetic triad; if the contacts do not lie on a conic they form an azygetic triad. Three tangents are syzygetic provided any one of the three is in the Steiner complex determined by the other two. We see that any three tangents of an Aronhold set are azygetic, and that these are the only sets of 7 having this property.

5. *Contact Cubics of J'.* Contact cubics of J' are the maps of invariant 9-ics with triple points at p_i and 6 other double points on J. Such 9-ics are not necessarily composite; the following is a list of all types which are composite:

(6)

a) $f_3(i_1i_2i_3i_4i_5i_6)\cdot f_6(i_1^2i_2^2i_3^2i_4^2i_5^2i_6^2k_1^3)$,

b) $f_4(i_1^2i_2^2k_1k_2k_3k_4k_5)\cdot f_5(i_1i_2k_1^2k_2^2k_3^2k_4^2k_5^2)$,

c) $f_1\cdot f_8(i_1^3i_2^3i_3^3i_4^3i_5^3i_6^3i_7^3)$,

d) $f_2(i_1i_2i_3)\cdot f_7(i_1^2i_2^2i_3^2k_1^3k_2^3k_3^3k_4^3)$,

e) $f_3(i_1^2k_1k_2k_3k_4)\cdot f_6(i_1i_2^3i_3^3k_1^2k_2^2k_3^2k_4^2)$,

f) $f_4(i_1^3k_1k_2k_3k_4k_5k_6)\cdot f_5(k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2)$,

g) $f_4(i_1i_2i_3k_1^2k_2^2k_3^2)\cdot f_5(i_1^2i_2^2i_3^2k_1k_2k_3k_4^3)$.

Of types a) and b) there are 7 and 21 quadratic systems of ∞^3 9-ics, and these map into 28 systems of contact cubics of J'. Of the remaining types there are 1, 35, 105, 7, and 140 quadratic systems of ∞^2 9-ics. We show now that these systems of ∞^2 9-ics are special systems included in systems of ∞^3 9-ics.

We know that the locus of vertices of the cones in a net of quadrics on 8 points s_i is a sextic S, of which each line s_is_k is a bisecant; S may be mapped upon a general plane quartic J', the 56 contacts of double tangents arising from the 56 points cut out of S by s_is_k, and the 6 points of any plane section of S map into the points of contact of a contact cubic of J'. It has been shown that by subjecting the plane sections of S to Cremona transformations whose base points are at the points s_i, we find additional systems of points which map into contacts of a contact cubic; in all there are 36 such systems

(including the original plane sections).* If we project S upon a plane Π from one of the 8 base points, say s_8, we obtain a sextic J, with 7 double points p_i, which is the locus of fixed points in a Geiser involution on Π. Each of the sets of 6 points on S projects into a set of 6 points on J which still correspond to the contacts of a contact cubic of J'. The line sections of J do not correspond to the general plane sections of S, but to those sections of S whose planes pass through s_8; hence, the types c) $\cdots$ g) are not the most general 9-ics which map into contact cubics of J', but are special systems. In fact, since each component of every 9-ic of these types is a rational curve, these 288 systems of ∞^2 9-ics map into the 288 systems of rational contact cubics of J'; 8 of these systems are included in each of 36 systems of ∞^3 contact cubics of J', as pointed out by Clebsch.† In order to see how the 288 systems of types c) $\cdots$ g) correspond to the 36 systems of ∞^3 contact cubics, we arrange these 288 systems in 36 groups of 8 systems each, such that the 36 groups are permuted imprimitively among themselves by Cremona transformations (in the plane Π) whose base points are at the points p_i. The first group contains type c) and the 7 systems of type f). One of the remaining 35 groups contains the 8 systems,

$$
\begin{aligned}
(7)\qquad & f_2(i_1i_2i_3)\cdot f_7(i_1^2i_2^2i_3^2k_1^3k_2^3k_3^3k_4^3),\\
& f_3(i_1^2k_1k_2k_3k_4)\cdot f_6(i_1i_2^3i_3^3k_1^2k_2^2k_3^2k_4^2),\\
& f_3(i_2^2k_1k_2k_3k_4)\cdot f_6(i_1^3i_2i_3^3k_1^2k_2^2k_3^2k_4^2),\\
& f_3(i_3^2k_1k_2k_3k_4)\cdot f_6(i_1^3i_2^3i_3k_1^2k_2^2k_3^2k_4^2),\\
& f_4(i_1i_2i_3k_2^2k_3^2k_4^2)\cdot f_5(i_1^2i_2^2i_3^2k_1^3k_2k_3k_4),\\
& f_4(i_1i_2i_3k_1^2k_3^2k_4^2)\cdot f_5(i_1^2i_2^2i_3^2k_1k_2^3k_3k_4),\\
& f_4(i_1i_2i_3k_1^2k_2^2k_4^2)\cdot f_5(i_1^2i_2^2i_3^2k_1k_2k_3^3k_4),\\
& f_4(i_1i_2i_3k_1^2k_2^2k_3^2\cdot\ f_5(i_1^2i_2^2i_3^2k_1k_2k_3k_4^3).
\end{aligned}
$$

By selecting $i_1i_2i_3$ in all possible ways from 1, 2, $\cdots$, 7 we find all 35 groups.

In the same way one may go on enumerating systems of contact curves of higher order. We shall not do this, but shall return to a further consideration of the Geiser involution in the x plane, showing how the contact curves of J' may be obtained without using the mapping system $y_i = C_i(x)$.

6. *A Correlation between the x and y Planes.* Any line L of the x

* Coble, *Transactions of the American Mathematical Society*, Vol. 17 (1916), p. 377.

† *Mathematische Annalen*, Vol. 3 (1871), p. 52.

plane is transformed by G into an octavic cut by L in 8 points, of which 6 are fixed points of G, lying on J. Thus L carries a single pair of copoints (P, Q) which are interchanged by G; the locus of this pair, as L turns about a point A, is the isologous curve corresponding to A; we denote it by C_A. If the equations defining G are $x'_i = (g_i x)^8$, $(i = 0, 1, 2)$ we see that the isologous curve corresponding to (a_0, a_1, a_2) is

$$\begin{vmatrix} a_0 & a_1 & a_2 \\ x_0 & x_1 & x_2 \\ (g_0x)^8 & (g_1x)^8 & (g_2x)^8 \end{vmatrix} = 0.$$

But J is a factor of this determinant, and therefore the isologous curves are the cubics of the net on p_i. Moreover, let us understand by C_0, C_1, C_2 those particular cubics which are the two rowed minors of

$$\left\| \begin{matrix} x_0 & x_1 & x_2 \\ (g_0x)^8 & (g_1x)^8 & (g_2x)^8 \end{matrix} \right\|$$

(after the removal of the common factor J). With this convention the isologue of (a_0, a_1, a_2) is $a_0C_0 + a_1C_1 + a_2C_2 = 0$. Since this involves no loss of generality, we shall suppose that C_i has this meaning in all the preceding discussions.

The lines of the plane on which the points P and Q coincide form a locus; there are 4 such lines through any point A, since from A there can be drawn 4 tangents to C_A. Hence these lines envelop a curve of class 4, denoted by E_4. Any point P of J has the property that all cubics of the net which pass through P do so in a fixed direction; if at the points of J we draw the lines in these directions, we have the lines of E_4. Thus E_4 is in (1, 1) correspondence with J, and is of genus 3; E_4 has, therefore, no flexes or double tangents, but has 24 cusps and 28 double points. If A is such that two of the four intersections of C_A with J coincide, then A is a point of E_4. Such a cubic C_A must have a double point on J; hence E_4 is the locus of the isologous centers of the nodal cubics of the net. If A is such that the four intersections of C_A with J form two coincident pairs, then A is a double point of E_4; there are 28 such points, namely, the 7 points p_i, and the isologous centers of the 21 composite cubics of the net. Similarly, we see that the 24 cusps of E_4 are the isologous centers of the 24 cubics of the net which have cusps.

Let $P = (x_0, x_1, x_2)$ and let Q be the copoint of P; the coordinates of the line PQ are

$$u_0 : u_1 : u_2 = \left\| \begin{matrix} x_0 & x_1 & x_2 \\ (g_0x)^8 & (g_1x)^8 & (g_2x)^8 \end{matrix} \right\| = C_0 : C_1 : C_2.$$

In § 2 we saw that any invariant curve C_{3k} may be written $(aC)^k = 0$; the locus of lines joining corresponding points of C_{3k} is $E_k = (au)^k = 0$; the map of C_{3k} in the y plane is $C'_k = (ay)^k = 0$. Therefore we have

THEOREM II. *The equations $u_i = y_i$ $(i = 0, 1, 2)$ define a correlation between the x and y planes such that to a curve C'_k in y corresponds the envelope E_k in x; the pairs of copoints (P, Q) on the lines of E_k form a curve C_{3k} which is the invariant curve corresponding to C'_k in the mapping $y_i = C_i(x)$.*

In particular, this correlation sends E_4 into J'. Thus the facts concerning Steiner complexes and Aronhold sets of double tangents of J' give corresponding facts concerning Steiner complexes and Aronhold sets of double points of E_4; for example, the points p_i form an Aronhold set.

7. *Perspective Curves.* We consider now the locus E_m of lines joining corresponding points of two projective rational curves R_n and $R_{n'}$. If the curves are expressed parametrically

$$R_n : X_i = (a_i t)^n,\ R_{n'} : X'_i = (a'_i t)^{n'},\quad i = 0, 1, 2$$

the coordinates of the line XX' are

$$U_0 : U_1 : U_2 = \begin{Vmatrix} (a_0 t)^n & (a_1 t)^n & (a_2 t)^n \\ (a'_0 t)^{n'} & (a'_1 t)^{n'} & (a'_2 t)^{n'} \end{Vmatrix} .$$

Therefore E_m is a rational envelope of class $m = n + n'$. Suppose that R_n and $R_{n'}$ meet at a point given by the same value of t on each curve, say, $t = t_0$; then U_0, U_1, U_2 have a common factor $t - t_0$, and the class of E_m is reduced by one; in general, the class of E_m is $m = n + n' - i$, where i is the number of values of t which give points of intersection of R_n and $R_{n'}$. The loci R_n and E_m are such the point t of R_n lies on the line t of E_m; any two curves having this property are said to be perspective. We note that E_m is perspective to $R_{n'}$ as well as to R_n. It has been shown * that a given rational curve R_n has ∞^{2m-n+1} perspective rational envelopes E_m which, when expressed parametrically, lie in a linear system. Each E_m touches R_n at $m + n - 2$ points.

We return now to the plane of the Geiser involution. The points of any line $L_1 = f_1(i_1)$ may be expressed linearly in terms of a parameter t; the transform of L_1 in G is $L_5 = f_5(i_1 k_1^2 k_2^2 k_3^2 k_4^2 k_5^2 k_6^2)$, which may be expressed

* Coble, *American Journal*, Vol. 32 (1910), p. 351.

rationally in terms of the same parameter t. L_1 and L_5 meet in 4 points given by the same values of t on each curve, namely, the 4 points in which L_1 cuts J (outside of p_i). The lines joining corresponding points of L_1 and L_5 form an envelope of class $1+5-4=2$; this conic E_2 is the unique perspective conic of L_5; it touches L_5 5 times and L_1 once. Moreover, E_2 touches E_4 at 4 points, the 4 common tangents (each counted twice) being the lines of E_4 corresponding to the 4 points of J cut out by L_1. As the quintic L_5 describes a pencil, the corresponding conic E_2 describes a quadratic system; we see this as follows: the invariant sextics $f_1(i_1)\cdot f_5(i_1k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2)$ form a quadratic system which may be written $(aC)^2(b\lambda)^2=0$; the lines joining corresponding points form the locus $(au)^2(b\lambda)^2=0$, which is a quadratic system of conics. We have therefore

THEOREM III. *If quintics of a pencil have six common double points and one simple point at seven points forming an Aronhold set of double points of a quartic envelope E_4, then the conics perspective to the pencil of quintics form a quadratic system, and are contact conics of E_4.*

The dual theorem could be stated for line quintics and double tangents of the quartic J'.

Consider similarly the system of invariant sextics

$$L_2\cdot L_4=f_2(i_1i_2i_3i_4)\cdot f_4(i_1i_2i_3i_4k_1^2k_2^2k_3^2).$$

The lines joining corresponding points of any one sextic again envelop a conic, which is that one of the ∞^1 conics perspective to L_4 and at the same time perspective to L_2. This conic envelope touches L_2 twice and L_4 4 times. As the sextic $L_2\cdot L_4$ describes a quadratic system, the perspective conic again describes a quadratic system of contact conics of E_4. By considering similarly all composite invariant sextics of the types given by (5), we find 63 systems of contact conics of E_4; moreover, we find in this way all contact conics. But there are 288 Aronhold sets of double points of E_4, from any one of which we might get all the contact conics; therefore each conic may be obtained in 288 ways. Any one conic arises $288\cdot 7/63=32$ times from a line and quintic, $288\cdot 35/63=160$ times from a conic and quartic, and $288\cdot 21/63=96$ times from a pair of cubics. Since each contact conic touches 6 times the sextic from which it arises, there are on each contact conic 288 sets of 6 contacts each; 32 of these sets divide into a single point and a set of 5, 160 divide into a pair and a set of 4, and 96 divide into two triads.

It is easily shown that the lines joining corresponding points of any com-

posite invariant curve, no component of which is invariant, form an envelope which is a contact locus of E_4. This involves perspective curves which are not rational. There is no general theory for this case; we point out only the fact that we can get from the Geiser involution systems of perspective curves of any genus. The corresponding contact envelopes of E_4 are the duals of the contact curves of J', as shown by Theorem II.

III. THE BERTINI INVOLUTION, B.

8. *Properties of B.* Let $p_1 \cdots p_8$ be arbitrary points of the x plane, and let O be the ninth base point of the pencil of cubics on p_i. Four linearly independent sextics with double points at p_i are C_0^2, C_0C_1, C_1^2, $L \cdot K$, where C_0 and C_1 are two cubics of the pencil on p_i, L is the line p_1p_2, and K the quintic simple at p_1p_2 and double at $p_3 \cdots p_8$. Moreover, there can not be more than four linearly independent sextics double at p_i, for if there were five linearly independent ones we could construct a pencil of them having L as a factor, which would imply the existence of a pencil of quintics K; but there can be no such pencil of quintics, for if there were, it could be reduced by a Cremona transformation to a pencil of conics on five points. There are therefore ∞^3 sextics double at p_i; the web may be written

$$a_0C_0^2 + 2a_1C_0C_1 + a_2C_1^2 + a_3S = 0. \tag{8}$$

Through any point P there will be a net of these sextics, which may be written

$$(k_0C_0 + k_1C_1)(l_0C_0 + l_1C_1) + l_2S = 0 \tag{9}$$

where $k_0C_0 + k_1C_1 = 0$ is the cubic of the pencil on p_i passing through P, and $S = 0$ is a sextic of the web (8) passing through P; (the parameters of the net are l_0, l_1, l_2). All sextics of the net (9) pass through the intersections of $k_0C_0 + k_1C_1 = 0$ and $S = 0$, of which 16 are at $p_1 \cdots p_8$, a seventeenth at P, and one remaining one at Q. Thus all sextics of the web which pass through a point P pass also through a point Q; P and Q are copoints in B. It is evident that all cubics on p_i, and all sextics (8) are invariant under B. Hence O is a self corresponding point.

Let S_i be the sextic of the web which has a triple point at p_i; if P is any point on S_i, the copoint Q is at p_i. Thus p_i is a fundamental point, the directions about p_i corresponding to the points of S_i.

Two lines L_1 and L_2 are transformed by B into curves B_1 and B_2, of

order m, with six fold points at p_i. Since B_1 and B_2 meet outside of p_i in a single point, we have $m^2 - 8 \cdot 36 = 1$, whence $m = 17$.

Since the cubic $k_0C_0 + k_1C_1 = 0$ is invariant, B determines an involution on this cubic. We know that on the elliptic cubic there are two types of involutions; in one type the elliptic parameters of corresponding points differ by half a period, and there are therefore no fixed points; in the second type the sum of the elliptic parameters of corresponding points is constant, i. e. pairs are cut out of the cubic by lines through a point T of the cubic; there are four fixed points, namely the four contacts of tangents from T. In the involution on $k_0C_0 + k_1C_1 = 0$ determined by B we know that O is a fixed point, whence the involution is of the second type, and pairs of corresponding points are collinear with T, where T is the tangential point of O with respect to $k_0C_0 + k_1C_1 = 0$. From this we see that O is not only a fixed point of B, but every direction about O is a fixed direction. Aside from O, there are three fixed points of B on $k_0C_0 + k_1C_1 = 0$; the locus of these triads is the fixed curve J. Suppose J is of order j, with k fold points at p_i; then the cubics of the pencil on p_i cut J in three variable points, whence $3j - 8k = 3$. Since the transform of J in B is of order j we have $17j - 8 \cdot 6k = j$. Then $j = 9$, $k = 3$, whence J is a 9-ic with triple points at p_i. It is evident that the three directions of J at p_i are the same as those of the fundamental sextic S_i. J is the locus of the ninth double point of a sextic of the web (8); the equation of J is

$$\frac{\partial(C_0,\ C_1,\ S)}{\partial(x_0,\ x_1,\ x_2)} = 0.$$

We recall the following theorem on the general cubic: *From any point O of the cubic there can be drawn four tangents, touching at $D_1 \cdots D_4$. The diagonal triangle $T_1T_2T_3$ of the quadrangle (D) has its vertices on the cubic; moreover the tangents at $T_1T_2T_3$ are concurrent (on the cubic) at T, the tangential of the point O.* Remembering that the points D_i are cut out of the cubic by the polar conic of the point O we have concerning B

THEOREM IV. *The polar conics of O with respect to the pencil of cubics on p_i form a pencil projective to the pencil of cubics; corresponding members of these pencils intersect in a quadruple of points D_i, of which the diagonal triangle is T_i. The locus of the triads T_i is the fixed curve J.*

It is easy to demonstrate the further facts; the locus of the quadruple D_i is a quintic C_5 simple at p_i and triple at O. The base points of the pencil

of polar conics of O (with respect to $k_0C_0 + k_1C_1 = 0$) are the point O and the three points cut out of C_5 by the tangents to C_5 at O. The locus of T (the tangential of O with respect to $k_0C_0 + k_1C_1 = 0$) is a quartic C_4 simple at p_i and triple at O; C_4 is the first polar of O with respect to C_5. The tangents to C_5 and C_4 at O are the flex tangents of the three cubics of the pencil which have flexes at O.

The isologous curve corresponding to a point P is a 9-ic with triple points at p_i, tangent to OP at O, and tangent to PQ at P; we denote it by N_P. At every point of J there are two fixed directions, one of which is not the direction of J itself; the locus of lines which cut J in these fixed directions (i. e. the locus of lines on which two copoints P and Q have come together) is an envelop E_9; its class is 9 because N_P cuts J in 9 points outside of p_i. From P there can be drawn, aside from OP, 21 tangents to N_P; the contacts of 9 of these lie on J. Therefore N_P has $\frac{1}{2}(21 - 9) = 6$ double tangents which pass through P. Through any point of J there is a pencil of curves N_P which have a common tangent, and therefore one curve having a double point; the isologous center of this particular 9-ic is a point of E_9. Thus E_9 is the locus of isologous centers of those curves N_P which have double points on J.

If P is a point of C_4 it is evident that the cubic determined by p_i and P is a component of N_P; hence

THEOREM V. *C_4 is the locus of a point whose isologous 9-ic is a cubic and a sextic, each an invariant curve.*

The isologous 9-ic corresponding to O is formed by the three cubics of the pencil having flexes at O. The isologous 9-ic corresponding to p_i is formed by the sextic S_i and the cubic of the pencil which touches Op_i at O.

An invariant curve with k fold points at p_i must be of order $3k$, and such a C_{3k} must pass through O an odd or even number of times according as k is odd or even. By counting the number of intersections of C_{3k} with N_P we find that in general the lines joining corresponding points of C_{3k} form an envelope of class $\frac{1}{2}(3k - \omega)$, where ω is the multiplicity of C_{3k} at O. For example, an invariant 9-ic will give an envelope of class 4; but if the 9-ic is an isologous curve the class reduces to one. This is due to the fact that two isologous curves N_P and $N_{P'}$ meet once at O, 8 times on PP', and nowhere else outside of p_i.

If C_{3k} is composite, and if no component is itself invariant, then C_{3k} cuts J in double points only, outside of p_i, and the corresponding envelope

E touches E_9 on the lines of E_9 corresponding to the double points of C_{3k} which lie on J. Thus, as in the Geiser involution, we are led to perspective envelopes E which have contact with E_9, but here the contacts do not account for all the common tangents of E and E_9. This is due to the fact that G is of class one and B of class four, where by class we mean the number of pairs of copoints lying on an arbitrary line. This increase in class makes the geometry of E_9 more difficult than that of E_4 (in G). For example, E_9, being of genus 4, must have 24 flexes and double tangents; if E_9 has d double tangents and $24-d$ flexes, we find from the Plücker formulas that E_9 is of order d with $\frac{1}{2}d(d-7)$ double points and $2d-3$ cusps. It is clear that E_9 has triple points at p_i, whence d must be at least 12, but the exact value of d has not been determined.

9. *A Mapping System; Contact Surfaces of J'.* The equations

$$y_0 = C_0^2, \quad y_1 = C_0C_1, \quad y_2 = C_1^2, \quad y_3 = S \tag{10}$$

map pairs of copoints upon points of a cone K whose equation is $y_0y_2 - y_1^2 = 0$. The point O maps into O', the vertex of K. Cubics of the pencil on p_i map into the lines of K, and invariant sextics map into plane sections. The fixed curve J maps into a sextic J' of genus 4. Since any curve on the cone is cut out by some surface $(ay)^n = 0$ we see that any invariant curve in x may be written $(ay)^n = 0$. This equation, when $C_0^2, \cdots$, are put in for $y_0, \cdots$, may contain extraneous factors, as in the case of an invariant 9-ic; this will happen whenever the curve on K is only a part of the intersection of K with $(ay)^n = 0$. In particular J' is cut out of K by $(ay)^3 = 0$, giving rise to an 18-ic in x which is the square of J; hence J^2, but not J itself, is expressible as a polynomial in y.

Using the method of § 2 we find that the map on K of a composite invariant curve, no component of which is invariant, is a contact curve of J'.

If an invariant sextic is such that its 6 intersections with J coincide in pairs, then the map is a conic cut out of K by a tritangent plane of J'. The 8 fundamental sextics and all composite invariant sextics are of this type; they may be written

(11)

a) $f_6(k_1^3 i_1 i_2 i_3 i_4 i_5 i_6 i_7)$,

b) $f_1(i_1 i_2) \cdot f_5(i_1 i_2 k_1^2 k_2^2 k_3^2 k_4^2 k_5^2 k_6^2)$,

c) $f_2(i_1 i_2 i_3 i_4 i_5) \cdot f_4(i_1 i_2 i_3 i_4 i_5 k_1^2 k_2^2 k_3^2)$,

d) $f_3(k_1^2 i_1 i_2 i_3 i_4 i_5 i_6) \cdot f_3(k_2^2 i_1 i_2 i_3 i_4 i_5 i_6)$.

Of those types there are 8, 28, 56, 28 sextics, respectively, which correspond to the 120 tritangent planes of J'.

Contact quadrics of J' cut K in quartic curves which are the maps of invariant 12-ics having 6 double points on J not at p_i. Such 12-ics need not be composite; those which are composite are of the following types:

(12)

a) $f_3(i_1i_2i_3i_4i_5i_6i_7)\cdot f_9(i_1^3i_2^3i_3^3i_4^3i_5^3i_6^3i_7^3k_1^4)$,

b) $f_4(i_1^2i_2^2k_1k_2k_3k_4k_5k_6)\cdot f_8(i_1^2i_2^2k_1^3k_2^3k_3^3k_4^3k_5^3k_6^3)$,

c) $f_5(i_1^2i_2^2i_3^2i_4^2i_5^2k_1k_2k_3)\cdot f_7(i_1^2i_2^2i_3^2i_4^2i_5^2k_1^3k_2^3k_3^3)$,

d) $f_6(i_1^3i_2k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2)\cdot f_6(i_1i_2^3k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2)$,

e) $f_1(i_1)\cdot f_{11}(i_1^3k_1^4k_2^4k_3^4k_4^4k_5^4k_6^4k^4_7)$,

f) $f_2(i_1i_2i_3i_4)\cdot f_{10}(i_1^3i_2^3i_3^3i_4^3k_1^4k_2^4k_3^4k_4^4)$,

g) $f_3(i_1^2k_1k_2k_3k_4k_5)\cdot f_9(i_1^2i_2^4i_3^4k_1^3k_2^3k_3^3k_4^3k_5^3)$,

h) $f_4(i_1i_2i_3i_4k_1^2k_2^2k_3^2)\cdot f_8(i_1^3i_2^3i_3^3i_4^3k_1^2k_2^2k_3^2k_4^4)$,

i) $f_4(i_1^3k_1k_2k_3k_4k_5k_6k_7)\cdot f_8(i_1k_1^3k_2^3k_3^3k_4^3k_5^3k_6^3k_7^3)$,

j) $f_5(i_1k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2)\ .\ f_7(i_1^3k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2k_7^4)$,

k) $f_5(i_1^2i_2^2i_3^2k_1^3k_2k_3k_4k_5)\cdot f_7(i_1^2i_2^2i_3^2k_1k_2^3k_3^3k_4^3k_5^3)$,

l) $f_6(i_1^3i_2^3i_3i_4k_1^2k_2^2k_3^2k_4^2)\cdot f_6(i_1i_2i_3^3i_4^3k_1^2k_2^2k_3^2k_4^2)$.

Of types a), b), c), d) there are 8, 28, 56, 28 quadratic systems of ∞^2 12-ics; these systems correspond to 120 systems of contact quadrics of J'. Of types e), f), $\cdots$, l) there are 8, 70, 168, 280, 8, 56, 280, 210 quadratic systems of ∞^1 12-ics. These 1080 systems map into rational contact quartic curves of J', and are therefore special systems contained in systems of ∞^2 12-ics. The 1080 systems can be arranged in 135 groups of 8 systems each, such that the groups are permuted imprimitively among themselves by all quadratic transformations with fundamental points at three of the points p_i. The grouping is as follows:

(13) The 8 systems of type i),

(14)

$f_1(i_1)\cdot f_{11}(i_1^3k_1^4k_2^4k_3^4k_4^4k_5^4k_6^4k_7^4)$,

$f_5(i_1k_2^2k_3^2k_4^2k_5^2k_6^2k_7^2)\cdot f_7(i_1^3k_1^4k_2^2k_3^2k_4^2k_5^2k_6^2k_7^2)$,

$f_5(i_1k_1^2k_3^2k_4^2k_5^2k_6^2k_7^2)\cdot f_7(i_1^3k_1^2k_2^4k_3^2k_4^2k_5^2k_6^2k_7^2)$,

$f_5(i_1k_1^2k_2^2k_4^2k_5^2k_6^2k_7^2)\cdot f_7(i_1^3k_1^2k_2^2k_3^4k_4^2k_5^2k_6^2k_7^2)$,

$f_5(i_1k_1^2k_2^2k_3^2k_5^2k_6^2k_7^2)\cdot f_7(i_1^3k_1^2k_2^2k_3^2k_4^4k_5^2k_6^2k_7^2)$,

$f_5(i_1k_1^2k_2^2k_3^2k_4^2k_6^2k_7^2)\cdot f_7(i_1^3k_1^2k_2^2k_3^2k_4^2k_5^4k_6^2k_7^2)$,

$f_5(i_1k_1^2k_2^2k_3^2k_4^2k_5^2k_7^2)\cdot f_7(i_1^3k_1^2k_2^2k_3^2k_4^2k_5^2k_6^4k_7^2)$,

$f_5(i_1k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2)\cdot f_7(i_1^3k_1^2k_2^2k_3^2k_4^2k_5^2k_6^2k_7^4)$.

(15) $f_2(i_1i_2i_3i_4)\cdot f_{10}(i_1^3i_2^3i_3^3i_4^3k_1^4k_2^4k_3^4k_4^4)$,
$f_4(i_1i_2i_3i_4k_2^2k_3^2k_4^2)\cdot f_8(i_1^3i_2^3i_3^3i_4^3k_1^4k_2^2k_3^2k_4^2)$,
$f_4(i_1i_2i_3i_4k_1^2k_3^2k_4^2)\cdot f_8(i_1^3i_2^3i_3^3i_4^3k_1^2k_2^4k_3^2k_4^2)$,
$f_4(i_1i_2i_3i_4k_1^2k_2^2k_4^2)\cdot f_8(i_1^3i_2^3i_3^3i_4^3k_1^2k_2^2k_3^4k_4^2)$,
$f_4(i_1i_2i_3i_4k_1^2k_2^2k_3^2)\cdot f_8(i_1^3i_2^3i_3^3i_4^3k_1^2k_2^2k_3^2k_4^4)$,
$f_6(i_1i_2i_3^3i_4^3k_1^2k_2^2k_3^2k_4^2)\cdot f_6(i_1^3i_2^3i_3i_4k_1^2k_2^2k_3^2k_4^2)$,
$f_6(i_1i_2^3i_3i_4^3k_1^2k_2^2k_3^2k_4^2)\cdot f_6(i_1^3i_2i_3^3i_4k_1^2k_2^2k_3^2k_4^2)$,
$f_6(i_1i_2^3i_3^3i_4k_1^2k_2^2k_3^2k_4^2)\cdot f_6(i_1^3i_2i_3i_4^3k_1^2k_2^2k_3^2k_4^2)$,

(16) $f_3(i_1^2k_1k_2k_3k_4k_5)\cdot f_9(i_1^2i_2^4i_3^4k_1^3k_2^3k_3^3k_4^3k_5^3)$,
$f_3(i_2^2k_1k_2k_3k_4k_5)\cdot f_9(i_1^4i_2^2i_3^4k_1^3k_2^3k_3^3k_4^3k_5^3)$,
$f_3(i_3^2k_1k_2k_3k_4k_5)\cdot f_9(i_1^4i_2^4i_3^2k_1^3k_2^3k_3^3k_4^3k_5^3)$,
$f_5(i_1^2i_2^2i_3^2k_1^3k_2k_3k_4k_5)\cdot f_7(i_1^2i_2^2i_3^2k_1k_2^3k_3^3k_4^3k_5^3)$,
$f_5(i_1^2i_2^2i_3^2k_1k_2^3k_3k_4k_5)\cdot f_7(i_1^2i_2^2i_3^2k_1^3k_2k_3^3k_4^3k_5^3)$,
$f_5(i_1^2i_2^2i_3^2k_1k_2k_3^3k_4k_5)\cdot f_7(i_1^2i_2^2i_3^2k_1^3k_2^3k_3k_4^3k_5^3)$,
$f_5(i_1^2i_2^2i_3^2k_1k_2k_3k_4^3k_5)\cdot f_7(i_1^2i_2^2i_3^2k_1^3k_2^3k_3^3k_4k_5^3)$,
$f_5(i_1^2i_2^2i_3^2k_1k_2k_3k_4k_5^3)\cdot f_7(i_1^2i_2^2i_3^2k_1^3k_2^3k_3^3k_4^3k_5)$,

There are 1, 8, 70, 56 groups of types (13), (14), (15), (16) which correspond to 135 systems of contact quadrics of J'.

In the same way one can go on enumerating systems of contact surfaces of higher order.

IV. The Jonquiere Involution, J_n.

10. *Properties of* J_n. Any curve J of order n with an $n-2$ fold point at O may be written $x_0^2(ax)^{n-2}+2x_0(bx)^{n-1}+(cx)^n=0$ by taking the point (1, 0, 0) at O. Any line through O cuts J in two points A and B; if P and Q are such that $(PQAB)=-1$, then P and Q are copoints in J_n. The curve J is the locus of fixed points. Every line through O is invariant. O is a fundamental point, the directions about O corresponding to the points of the first polar of O with respect to J. The $2(n-1)$ points p_i, the contacts of tangents to J drawn from O, are also fundamental points, the directions about p_i corresponding to the points of the line Op_i. An arbitrary line is transformed by J_n into a curve of order n simple at p_i and $n-1$ fold at O.

We shall mean by H_m any curve of order m passing once through each point p_i, and $m-2$ times through O. By $f_m(0^{e}1^{e_1}2^{e_2}\cdots)$ we mean any curve

of order m with multiplicity e at O and e_i at p_i. We divide the Jonquiere involutions into two classes, according as they are of odd or even order.

11. *Case I:* $n = 2k - 1$. The $4(k-1)$ points p_i form the base points of a pencil of curves H_k, each of which is invariant under J_{2k-1};* this pencil may be written:

$$\mu_0 M_0 + \mu_1 M_1 = 0. \tag{17}$$

The pencil of lines through O may be written:

$$\lambda_0 L_0 + \lambda_1 L_1 = 0. \tag{18}$$

Then every curve of the system

$$a_0 L_0 M_0 + a_1 L_0 M_1 + a_2 L_1 M_0 + a_3 L_1 M_1 = 0 \tag{19}$$

is invariant.

The equations

$$y_0 = L_0 M_0, \quad y_1 = L_0 M_1, \quad y_2 = L_1 M_0, \quad y_3 = L_1 M_1 \tag{20}$$

map pairs of copoints in x upon points of the quadric $K = y_0 y_3 - y_1 y_2 = 0$. To a curve of system (17), (18), or (19) corresponds a μ generator, a λ generator, or a plane section of K. Each point p_i maps into a λ generator. The point O maps into a curve $(a\lambda)^{k-2}(b\mu)^1 = 0$. The fixed curve J maps into a curve J' whose equation is $(a\lambda)^{2k-2}(b\mu)^2 = 0$; hence the corresponding curve in x has the equation $(a'L)^{2k-2}(b'M)^2 = 0$ which is the square of J. Therefore J^2, but not J itself, is expressible as a form in L and M.

Since any invariant curve in x corresponds to some curve on K and conversely, any invariant curve may be written $(aL)^l(bM)^m = 0$. Therefore the general mapping by (2) is equivalent to a mapping from x to the quadric K, followed by a rational mapping of K upon the surface z.

The argument of § 2, applied to J_{2k-1} shows that a composite invariant curve, no component of which is invariant, maps into a contact curve of J'.

12. *Two transformations of Case I:* $n = 3$ and $n = 5$. If $k = 2$, J_{2k-1} is the cubic involution with four simple points p_i. The elliptic cubic J maps into an elliptic quartic J'. There are four lines of each regulus which touch J'; they are the maps of the four lines $f_1(0\, i_1)$, the three line pairs $f_1(i_1 i_2) \cdot f_1(k_1 k_2)$, and the conic $f_2(0\, i_1 i_2 i_3 i_4)$.

Contact conics of J' are maps of composite cubics, which must be of the

* Bateman, *Quarterly Journal*, Vol. 37, p. 278.

form $f_1(i_1) \cdot f_2(0\, k_1k_2k_3)$. There are four quadratic systems of ∞^1 cubics of this type, corresponding to the four systems of bitangent planes of J'.

Contact quadrics touching J' at 4 points cut K in elliptic quartics which are maps of invariant sextics double at O and p_i, with four other double points on J. Such sextics need not be composite; those which are composite are of the following types:

(21)
$$\begin{aligned} &\text{a)}\quad f_3(0\, i_1i_2i_3i_4) \cdot f_3(0\, i_1i_2i_3i_4),\\ &\text{b)}\quad f_2(i_1i_2) \cdot f_4(0^2\, i_1i_2k_1^2k_2^2),\\ &\text{c)}\quad f_3(0\, i^2{}_1k_1k_2) \cdot f_3(0\, i_2^2k_1k_2). \end{aligned}$$

Quartics which are the maps of type a) touch J' in four points which lie in a plane, as we shall now show. The cubic $f_3(0\, i_1i_2i_3i_4)$ is the general cubic on O and p_i, and may be written

$$a_0L_0M_0 + a_1L_0M_1 + a_2L_1M_0 + a_3L_1M_1 + a_4J = 0.$$

The transform of this cubic in J_3 is

$$a_0L_0M_0 + a_1L_0M_1 + a_2L_1M_0 + a_3L_1M_1 - a_4J = 0.$$

The invariant sextic of type a) is therefore

$$(a_0L_0M_0 + a_1L_0M_1 + a_2L_1M_0 + a_3L_1M_1)^2 - (a_4J)^2 = 0.$$

But this sextic cuts J at the same points as

$$a_0L_0M_0 + a_1L_0M_1 + a_2L_1M_0 + a_3L_1M_1 = 0$$

and these points map into a plane section of J'.

The 12 systems of ∞^3 sextics of types b) and c) can be arranged in three groups of four systems each, such that these groups are permuted imprimitively among themselves by all quadratic transformations with one fundamental point at O and two others at two of the points p_i. One such group is

(22)
$$\begin{aligned} &f_2(i_1i_2) \cdot f_4(0^2\, i_1i_2k_1^2k_2^2),\\ &f_2(k_1k_2) \cdot f_4(0^2\, k_1k_2i_1^2i_2^2),\\ &f_3(0\, i_1^2k_1k_2) \cdot f_3(0\, i_2^2k_1k_2),\\ &f_3(0\, k_1^2i_1i_2) \cdot f_3(0\, k_2^2i_1i_2). \end{aligned}$$

The three groups are found by pairing off the numbers 1, 2, 3, 4 in the three possible ways. These three groups correspond to the three systems of contact quadrics of J' associated with the proper half periods of the elliptic parameter which can be put on J'.

If $k = 3$, J_{2k-1} is the quintic involution with eight base points p_i which with O form the base points of a pencil of invariant cubics. The map of J is an hyperelliptic sextic J' with characteristic [4, 2].

Tritangent planes of J' cut K in conics which are maps of invariant quartics double at O and simple at p_i, with three other double points on J. The only quartics of this sort are

$$(23) \qquad \begin{aligned} &\text{a)} \quad f_1(i_1 i_2) \cdot f_3(0^2 k_1 k_2 k_3 k_4 k_5 k_6), \\ &\text{b)} \quad f_2(0\, i_1 i_2 i_3 i_4) \cdot f_2(0\, k_1 k_2 k_3 k_4). \end{aligned}$$

The point O also maps into a conic having three contacts with J'. Hence we have isolated the $28 + 35 + 1 = 64$ tritangent planes of J'.

Contact quadrics of J' cut K in $[2, 2]$ quartics which are maps of invariant curves of the form $f_8(0^4 i_1^2 i_2^2 i_3^2 i_4^2 i_5^2 i_6^2 i_7^2 i_8^2)$ with six other double points on J. Such octavics need not be composite; those which are composite are of the types:

$$(24) \qquad \begin{aligned} &\text{a)} \quad f_3(0\, i_1 i_2 i_3 i_4 i_5 i_6) \cdot f_5(0^3 i_1 i_2 i_3 i_4 i_5 i_6 k_1^2 k_2^2), \\ &\text{b)} \quad f_4(0^2 k_1^2 i_1 i_2 i_3 i_4 i_5 i_6) \cdot f_4(0^2 k_2^2 i_1 i_2 i_3 i_4 i_5 i_6), \\ &\text{c)} \quad f_2(i_1 i_2 i_3 i_4) \cdot f_6(0^4 i_1 i_2 i_3 i_4 k_1^2 k_2^2 k_3^2 k_4^2), \\ &\text{d)} \quad f_3(0\, i_1 i_2 i_3 i_4 k_1^2) \cdot f_5(0^3 i_1 i_2 i_3 i_4 k_2^2 k_3^2 k_4^2), \\ &\text{e)} \quad f_4(0^2 i_1 i_2 i_3 i_4 k_1^2 k_2^2) \cdot f_4(0^2 i_1 i_2 i_3 i_4 k_3^2 k_4^2). \end{aligned}$$

Types a) and b) correspond to $28 + 28 = 56$ systems of ∞^2 contact quadrics of J'. Types c), d), e) give 70, 280, 210 systems of ∞^1 quadrics. These 560 systems we arrange in 70 groups of 8 each, such that the groups are permuted imprimitively by quadratic transformations with one base point at O and two others at two of the points p_i. One such group is

$$(25) \qquad \begin{aligned} &f_2(i_1 i_2 i_3 i_4) \cdot f_6(0^4 i_1 i_2 i_3 i_4 k_1^2 k_2^2 k_3^2 k_4^2), \\ &f_3(0\, i_1 i_2 i_3 i_4 k_1^2) \cdot f_5(0^3 i_1 i_2 i_3 i_4 k_2^2 k_3^2 k_4^2), \\ &f_3(0\, i_1 i_2 i_3 i_4 k_2^2) \cdot f_5(0^3 i_1 i_2 i_3 i_4 k_1^2 k_3^2 k_4^2), \\ &f_3(0\, i_1 i_2 i_3 i_4 k_3^2) \cdot f_5(0^3 i_1 i_2 i_3 i_4 k_1^2 k_2^2 k_4^2), \\ &f_3(0\, i_1 i_2 i_3 i_4 k_4^2) \cdot f_5(0^3 i_1 i_2 i_3 i_4 k_1^2 k_2^2 k_3^2), \\ &f_4(0^2 i_1 i_2 i_3 i_4 k_1^2 k_2^2) \cdot f_4(0^2 i_1 i_2 i_3 i_4 k_3^2 k_4^2), \\ &f_4(0^2 i_1 i_2 i_3 i_4 k_1^2 k_3^2) \cdot f_4(0^2 i_1 i_2 i_3 i_4 k_2^2 k_4^2), \\ &f_4(0^2 i_1 i_2 i_3 i_4 k_1^2 k_4^2) \cdot f_4(0^2 i_1 i_2 i_3 i_4 k_2^2 k_3^2). \end{aligned}$$

These 70 grups, together with types (24), a) and b), correspond to 126 systems of contact quadrics of J'. We note further that these 126 systems can

be arranged in 63 pairs of systems. We pair together the systems of types a) and b) which make the same choice of k_1 and k_2; this gives 28 pairs. With the group (25) we pair the group obtained by interchanging i and k in (25); this gives the remaining 35 pairs.

13. *Case II:* $n = 2k$. The $2(2k-1)$ points p_i lie on a unique H_k, and on these points is a net of curves H_{k+1}, each of which is invariant under J_{2k};* this net may be written

$$a_0 H_{k+1} + (a_1 x_1 + a_2 x_2) H_k = 0. \tag{26}$$

The equations,

$$y_0 = H_{k+1}, \quad y_1 = x_1 H_k, \quad y_2 = x_2 H_k, \tag{27}$$

map pairs of copoints in x upon points of the plane y; to the net (26) in x corresponds the net of lines in y. Every point of H_k maps into $O' = (1, 0, 0)$ in y; each point p_i maps into a line through O', and O maps into a curve of order $k-1$ with $k-2$ branches at O'. The fixed curve J maps into J', of order $2k$, with $2k-2$ branches at O'. The $2(2k-1)$ tangents to J' from O' are the maps of the $2(2k-1)$ points p_i.

Since every invariant curve in x corresponds to some curve $(ay)^n = 0$, every invariant curve can be written in this form. After the value of y_i as given by (27) has been substituted in $(ay)^n = 0$ the equation may contain an extraneous factor H_k. For example, the equation of J' gives rise to an equation of order $2k(k+1)$ in x containing the factor H_k^{2k-2}; after the removal of this factor we have left an equation of order $4k$, which is the equation of J^2.

Since the general invariant curve is $(ay)^n = 0$ we see that the general mapping obtained from (2) is equivalent to the mapping (27) followed by a rational mapping of the y plane.

An argument similar to that of the § 2 shows that a composite invariant curve C, no component of which is invariant, maps into C', a contact curve of J'. Some of the intersections of C' and J' may fall at O', without any tangencies at O', but C' and J' touch at every common point aside from O'.

14. *A Transformation of Case II:* $n = 4$. If $k = 2$, J_{2k} is the quartic involution with a triple point at O and six simple points p_i. The six points p_i lie on a conic H_2 which maps into the point O'. J maps into a quartic J' with a double point at O'. The 16 double tangents of J' are the maps of the point O, and the 15 invariant cubics $f_1(i_1 i_2) \cdot f_2(0\, k_1 k_2 k_3 k_4)$.

* Bateman, *Quarterly Journal*, Vol. 37, p. 278.

Invariant quartics of the type

(28) a) $f_1(i_1) \cdot f_3(0^2 k_1 k_2 k_3 k_4 k_5)$,
b) $f_2(0\, i_1 i_2 i_3) \cdot f_2(0\, k_1 k_2 k_3)$

map into conics through O' and touching J' at three points; J' has $6+10=16$ quadratic systems of ∞^1 contact conics of this kind. Invariant sextics of types

(29) a) $f_2(i_1 i_2 i_3 i_4) \cdot f_4(0^2 i_1 i_2 i_3 i_4 k_1^2 k_2^2)$,
b) $f_3(0\, i_1 i_2 i_3 i_4 k_1^2) \cdot f_3(0\, i_1 i_2 i_3 i_4 k_2^2)$

map into conics touching J' at four points. The corresponding $15+15=30$ systems of contact conics form 15 pairs of systems; we pair together the types a) and b) which make the same choice of k_1 and k_2.

The systems of contact conics of type (29) have the property that each of the 30 systems contains 4 pairs of proper double tangents; and furthermore each of the 15 pairs of systems contains 8 pairs of proper double tangents, and these 8 pairs make up the entire set of 16. For example, let $k_1, k_2 = 1, 2$; then the system corresponding to (29), a) contains the proper line pairs

$$D_{12}D_{78}, \quad D_{34}D_{56}, \quad D_{35}D_{46}, \quad D_{36}D_{45};$$

the system corresponding to (29), b) contains the pairs

$$D_{13}D_{23}, \quad D_{14}D_{24}, \quad D_{15}D_{25}, \quad D_{16}D_{26}.$$

These results may be drawn from the theory of the theta functions for $p=2$.*

* Clebsch, *Leçons sur la Géométrie*, Vol. 3, pp. 266-286.

Algebraic and Transcendental Equations Connected with the Form of Stream Lines.

By H. Bateman.

Summary. When a departure from steady rectilinear flow is produced locally by a number of sources, sinks and vortices, an interesting type of transcendental equation gives the point or points in which a given stream line meets a given line. This equation is studied with the aid of a generalization of Koshliakov's theorem and a further result is obtained by varying the parameter of the stream line. Additional results relating to the form of stream lines are obtained by using conformal transformations and by studying the properties of the Jacobian of the component velocities.

1. *Extension of Koshliakov's theorem.**

Let

$$\phi_0 + i\psi_0 = \sum_{s=1}^{n} m_s \log(z - z_s) - \sum_{s=1}^{n} k_s \log(z - \zeta_s) \tag{1}$$

be the complex potential of the two dimensional flow produced by a number of sources and sinks, the sources being all above the axis of x and the sinks all on or below the axis of x. For the sake of greater generality m and k are supposed to be complex so that each source and sink has an associated vortex. Writing

$$z_s = a_s + ib_s, \qquad \zeta_s = a_s - i\beta_s, \qquad m_s = \lambda_s + i\sigma_s, \qquad k_s = \mu_s + i\tau_s$$

where a_s, b_s, a_s, β_s, λ_s, σ_s, μ_s, τ_s are all real, we have the inequalities

$$b_s > 0, \qquad \beta_s \geq 0, \qquad \lambda_s > 0, \qquad \mu_s > 0.$$

Now suppose that when x is real

$$\prod_{s=1}^{n} \frac{(x - z_s)^{\lambda_s + i\sigma_s}}{(x - \zeta_s)^{\mu_s + i\tau_s}} = f(x) + ig(x) \tag{2}$$

where $f(x)$ and $g(x)$ are real when x is real. If we superpose on the flow

* The case in which $m_s = 1$, $k_s = 0$ is considered by N. S. Koshliakov, *Messenger of Mathematics*, Vol. 55 (1926), p. 132, without any hydrodynamical interpretation.

produced by the sources and sinks a rectilinear flow specified by the stream-function $\psi_1 = x - y\tan\omega$ the stream function of the total flow is

$$\psi = \psi_0 + \psi_1$$

and the points in which a stream line $\psi = \theta$ cuts the axis of x are given by the transcendental equation

$$\tan^{-1}\frac{g(x)}{f(x)} + x = \theta$$

or
$$g(x)\cos(x-\theta) + f(x)\sin(x-\theta) = 0 \tag{3}$$

We wish to ascertain in the first place if the roots of this equation are all real. Writing

$$G(x) + iF(x) = e^{i(x-\theta)}\prod_{s=1}^{n}\frac{(x-a_s-ib_s)^{\lambda_s+i\sigma_s}}{(x-a_s+i\beta_s)^{\mu_s+i\tau_s}}$$

$$G(x) - iF(x) = e^{-i(x-\theta)}\prod_{s=1}^{n}\frac{(x-a_s+ib_s)^{\lambda_s-i\sigma_s}}{(x-a_s-i\beta_s)^{\mu_s-i\tau_s}}$$

we have

$$G(x) + iF(x) = e^{i(x-\theta)}[f(x) + ig(x)]$$

$$G(x) - iF(x) = e^{-i(x-\theta)}[f(x) - ig(x)]$$

$$F(x) = f(x)\sin(x-\theta) + g(x)\cos(x-\theta) \tag{4}$$

$$G(x) = f(x)\cos(x-\theta) - g(x)\sin(x-\theta).$$

Hence, if $z = x + iy$ is a root of the equation $F(z) = 0$ we have for this root

$$e^{i(z-\theta)}\prod_{s=1}^{n}\frac{(z-a_s-ib_s)^{\lambda_s+i\sigma_s}}{(z-a_s+i\beta_s)^{\mu_s+i\tau_s}} = e^{-i(z-\theta)}\prod_{s=1}^{n}\frac{(z-a_s+ib_s)^{\lambda_s-i\sigma_s}}{(z-a_s-i\beta_s)^{\mu_s-i\tau_s}}$$

Now

$$M_1 \equiv \left| e^{i(z-\theta)}\prod_{s=1}^{n}\frac{(z-a_s-ib_s)^{\lambda_s+i\sigma_s}}{(z-a_s+i\beta_s)^{\mu_s+i\tau_s}} \right|$$

$$= e^{-(y+\eta)}\prod_{s=1}^{n}\frac{[(x-a_s)^2+(y-b_s)^2]^{\lambda_s/2}}{[(x-a_s)^2+(y+\beta_s)^2]^{\mu_s/2}}$$

$$M_2 \equiv \left| e^{-i(z-\theta)}\prod_{s=1}^{n}\frac{(z-a_s+ib_s)^{\lambda_s-i\sigma_s}}{(z-a_s-i\beta_s)^{\mu_s-i\tau_s}} \right|$$

$$= e^{y+\eta'}\prod_{s=1}^{n}\frac{[(x-a_s)^2+(y+b_s)^2]^{\lambda_s/2}}{[(x-a_s)^2+(y-\beta_s)^2]^{\mu_s/2}}$$

where

$$\eta = \sum_{s=1}^{n}\left[\sigma_s\tan^{-1}\frac{y-b_s}{x-a_s} - \tau_s\tan^{-1}\frac{y+\beta_s}{x-a_s}\right]$$

$$\eta' = \sum_{s=1}^{n}\left[\sigma_s\tan^{-1}\frac{y+b_s}{x-a_s} - \tau_s\tan^{-1}\frac{y-\beta_s}{x-a_s}\right]$$

When there are no vortices $\eta = \eta' = 0$ and it is clear that if $y \neq 0$ $M_1 \neq M_2$ for when $y > 0$, $M_1 < M_2$ and when $y < 0$, $M_1 > M_2$. Hence when there are no vortices $y = 0$ and the roots of the equation (3) are all real.

A similar result may be obtained in some cases when there are vortices. If, for instance, we have

$$\sigma_s = \tau_s, \qquad a_s = a_s, \qquad b_s = \beta_s$$

whenever there are vortices associated with sources and sinks or isolated we have

$$\eta + \eta' = 0$$

and so

$$M_1 \neq M_2 \text{ if } y \neq 0$$

2. *The effect of varying* θ.

If x is a real root of the equation $F(x) = 0$ we have

$$dx/d\theta\,[f'(x)\sin(x-\theta) + g'(x)\cos(x-\theta) + G(x)] = G(x).$$

Now

$$\frac{\cos(x-\theta)}{f(x)} = \frac{\sin(x-\theta)}{-g(x)} = \frac{1}{G(x)}$$

$$\therefore \quad dx/d\theta\,[f(x)g'(x) - f'(x)g(x) + \{G(x)\}^2] = \{G(x)\}^2$$

But

$$\frac{f'(x)+ig'(x)}{f(x)+ig(x)} = \sum_{s=1}^{n}\left[\frac{\lambda_s+i\sigma_s}{x-a_s-ib_s} - \frac{\mu_s+i\tau_s}{x-a_s+i\beta_s}\right]$$

$$\frac{f'(x)-ig'(x)}{f(x)-ig(x)} = \sum_{s=1}^{n}\left[\frac{\lambda_s-i\sigma_s}{x-a_s+ib_s} - \frac{\mu_s-i\tau_s}{x-a_s-i\beta_s}\right]$$

$$\frac{f(x)g'(x)-f'(x)g(x)}{[f(x)]^2+[g(x)]^2} = \sum_{s=1}^{n}\left[\frac{\lambda_s b_s+\sigma_s(x-a_s)}{(x-a_s)^2+b_s^2} + \frac{\mu_s b_s-\tau_s(x-a_s)}{(x-a_s)^2+\beta_s^2}\right] \cdots (5)$$

If $\sigma_s = \tau_s = 0$ the right hand side is clearly positive and so $dx/d\theta$ is positive for all real values of θ. This means that when θ increases the point where the stream line meets the axis of x moves to the right (i. e. the direction in which x increases). If we increase θ by $\pi/2$, $F(x)$ is transformed into $-G(x)$ and if we add another $\pi/2$ to x the function $-G(x)$ is transformed into $-F(x)$, consequently we have a suggestion * that the roots of $F(x)$ are

* At first I was under the impression that this theorem could be proved with the aid of this remark but a conversation with Mr. C. B. Millikan has led me to adopt Koshliakov's method of proof.

separated by those of $G(x)$. To prove this theorem we adopt Koshliakov's method for the case $\theta = 0$ and calculate the derivative

$$\frac{d}{dx}\frac{F(x)}{G(x)} = \frac{f(x)g'(x) - f'(x)g(x) + [F(x)]^2 + [G(x)]^2}{G(x)^2}$$

When $\sigma_s = \tau_s = 0$ this is clearly positive and so the roots of $F(x) = 0$ are separated by those of $G(x) = 0$. It should be noticed that we also have the equation

$$\frac{d\theta}{dx} = \frac{d}{dx}\frac{F(x)}{G(x)}$$

where θ is defined by the equation $F(x) = 0$.

If we have two equations of the present type, viz.

$$F(x) = f(x)\sin(x-\theta) + g(x)\cos(x-\theta) = 0$$
$$F_1(x) = f_1(x)\sin(x-\theta) + g_1(x)\cos(x-\theta) = 0$$

and we multiply the left hand sides we obtain an equation of type

$$(gg_1 - ff_1)\cos 2(x-\theta) + (fg_1 + f_1g)\sin 2(x-\theta) + ff_1 + gg_1 = 0$$

whose roots are all real and increase with θ. We thus have one theorem relating to the roots of an equation of type

$$p(x)\cos 2(x-\theta) + q(x)\sin 2(x-\theta) + r(x) = 0 \quad \cdots (6)$$

An equation of this type arises also when we seek the condition that the equation

$$[f(x) + \lambda f_1(x)]\sin(x-\theta) + [g(x) + \lambda g_1(x)]\cos(x-\theta) = 0 \quad \cdots (7)$$

may have equal roots for some value of λ. Differentiating with respect to x and eliminating λ from the two equations we find that a double root x must satisfy the equation

$$[ff'_1 - f'f_1 - (gg'_1 - g'g_1)]\cos 2(x-\theta)$$
$$+ [f'g_1 - fg'_1 + f_1g' - f'_1g]\sin 2(x-\theta) = fg_1 - f_1g.$$

If this equation has no real root there will be no value of λ for which the equation (7) has two equal real roots. In particular, if

$$f_1(x) = g(x), \qquad g_1(x) = -f(x)$$

the equation becomes

$$[f(x)]^2 + [g(x)]^2 = 0$$

and this equation clearly has no real root. Since the equations

$$f(x)\sin(x-\theta)+g(x)\cos(x-\theta)=0$$
$$g(x)\sin(x-\theta)-f(x)\cos(x-\theta)=0$$

have only real roots we infer that the equation

$$[f(x)+\lambda g(x)]\sin(x-\theta)+[g(x)-\lambda f(x)]\cos(x-\theta)=0$$

has only real roots.* This may be verified by writing $\lambda=\tan\epsilon$ when the equation takes the form

$$f(x)\sin(x-\theta-\epsilon)+g(x)\cos(x-\theta-\epsilon)=0$$

and when ϵ is real this equation is of the type already considered and has all its roots real.

This example indicates that there may be other equations of type (7) which have only real roots for all real values of λ. In this connection it may be pointed out that the roots of the equation (6) are certainly not real when the inequality

$$[r(x)]^2>[p(x)^2]+[q(x)]^2$$

is satisfied for all values of x, for if we write

$$s(x)\,p(x)=-r(x)\cos 2\omega(x),\quad s(x)\,q(x)=-r(x)\sin 2\omega(x)$$

the equation then takes the form

$$\cos 2[x-\theta-\omega(x)]=s(x),\quad \text{where } |s(x)|>1$$

and this cannot be true for real values of x.

An equation of type

$$p(x)\cos x+q(x)\sin x+r(x)=0$$

whose roots are all real occurs in the theory of integral equations.†

Let $v(s)$ be the solution of the integral equation

$$u(s)=v(s)-\lambda\int_0^1\kappa(s,t)v(t)\,dt$$

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* As λ varies complex roots can appear only if 2 real roots come together and become complex. Hence if a double root never occurs the roots cannot become complex.

† H. Bateman, *Cambr. Phil. Trans.*, Vol. 20 (1908), p. 371. See also B. M. Wilson, *Proc. London Math. Soc.* (2), Vol. 24 (1924), p. 59.

where $k(s,t)$ is a real symmetric continuous function of s and t within the square $0 \leq s \leq 1$, $0 \leq t \leq 1$, then if

$$w(\lambda) = \int_0^1 u(s)v(s)\,ds$$
$$d/d\lambda\,[\lambda w(\lambda)] = \int_0^1 [v(s)]^2\,ds$$

and so the roots and poles of the function $\lambda w(\lambda) - c$ occur alternately as λ increases. Writing

$$\kappa(s,t) = \begin{matrix} s(1-t) & s \leq t \\ t(1-s) & s \geq t \end{matrix}$$

and taking $u(s)$ to be a polynomial of degree n in s the equation for λ is found to be

$$\begin{aligned} &\sin x\,[\textstyle\int_0^1 uu''ds - 1/x^2 \int_0^1 uu''''ds + 1/x^4 \int_0^1 uu^{(6)}ds - \cdots - c] \\ &\quad + x[u(0) - 1/x^2\ u''(0) + \cdots]\,[u(1) - 1/x^2\ u''(1) + \cdots] \\ &\quad - x\cos x\,[u(1) - 1/x^2\ u''(1) + 1/x^4\ u''''(1) - \cdots]^2 \\ &\quad + \sin x\,[u(1) - 1/x^2\ u''(1) + \cdots]\,[u'(1) - 1/x^2\ u'''(1) + \cdots] = 0 \end{aligned}$$

where $x^2 = \lambda$. If c is positive the roots of this equation in x are all real but if c is negative the equation in λ has a negative root and the equation in x a pair of imaginary roots. When $u = 1$ the equation is simply

$$x\cos x + c\sin x = x$$

and can be reduced to the homogeneous form

$$x\sin x/2 = c\cos x/2 \quad \text{or} \quad \sin x/2 = 0.$$

When $u(s) = a + bs$ the equation is

$$(a+b)^2\,x\cos x - [b(a+b) - c]\sin x = a(a+b)\,x$$

When c is negative and $\kappa(s,t)$ has the special value chosen above the roots of the equation $w(\lambda) = c$ are all real and positive and so the roots of the equation

$$\begin{aligned} &\sin x\,[\textstyle\int_0^1 uu''ds - 1/x^2 \int_0^1 uu''''ds + 1/x^4 \int_0^1 uu^{(6)}ds - \cdots - cx^2] \\ &\quad + x\,[u(0) - 1/x^2\,u''(0) + \cdots]\,[u(1) - 1/x^2\,u''(1) + \cdots] \\ &\quad - x\cos x\,[u(1) - 1/x^2\,u''(1) + 1/x^4\,u''''(1) - \cdots]^2 \\ &\quad + \sin x\,[u(1) - 1/x^2\,u''(1) + \cdots]\,[u'(1) - 1/x^2\,u'''(1) + \cdots] = 0 \end{aligned}$$

are all real. In particular, when $u(s) = a + bs$ we find that the roots of the equation

$$(a+b)^2 x \cos x - [b(a+b) - cx^2] \sin x = a(a+b)x$$

are all real.

When there are vortices present the quantity on the right of (5) is clearly positive in the following cases

1°. When $a_s = a_s$, $b_s = \beta_s$, $\sigma_s = \tau_s$.

In this case the vortices may be arranged in pairs which are images of each other in the axis of x.

2°. When $\sigma_s > 0$, $\tau_s < 0$, $x > a_s$, $x > a_s$.

In this case all the vortices are of one sign and we may conclude that x increases with θ when the stream line is to the right of all the vortices.

3°. When $\sigma_s < 0$, $\tau_s > 0$, $x < a_s$, $x < a_s$.

In this case x increases with θ when the stream line is on the left of all the vortices.

To show that if the vortices are unrestricted x does not always increase with θ let us consider the case of a single vortex. The equation of a stream line is then

$$\psi = x - y \tan \omega + \sigma \log [(x-a)^2 + (y-b)^2]^{\frac{1}{2}} = \theta$$

and when $y = 0$

$$x + \sigma \log [(x-a)^2 + b^2]^{\frac{1}{2}} = \theta$$

$$dx/d\theta \left[1 + \frac{\sigma(x-a)}{(x-a)^2 + b^2} \right] = 1.$$

If $\sigma^2 > 4b^2$ the quantity within square brackets can have either sign since it vanishes when

$$(x-a)^2 + \sigma(x-a) + b^2 = 0.$$

If q denotes the velocity along a stream line we have

$$q = \partial\psi/\partial n = \partial\psi/\partial x \cos \epsilon + \partial\psi/\partial y \sin \epsilon$$

$$0 = \partial\psi/\partial s = \partial\psi/\partial x \sin \epsilon - \partial\psi/\partial y \cos \epsilon$$

where ϵ denotes the angle which the normal to the stream line makes with the axis of x and $\partial/\partial s$, $\partial/\partial n$ denote tangential and normal derivatives respectively. We thus have

$$\partial\theta/\partial x = \partial\psi/\partial x = q\cos\epsilon$$

and so x increases with θ so long as the angle ϵ is an acute angle. In the example just considered some of the stream lines coming from infinity are caught in the vortex while others pass by the vortex and go to infinity in the direction of the stream. In the case of a stream line which is caught it is possible for the angle ϵ to be obtuse.

An obtuse angle can occur only when a stream line meets the axis of x in more than one point. When the equation for x has complex roots it means that a stream line has been caught by a vortex before it reached the axis of x. When there are no vortices the reason why our equation for x has only real roots becomes clear if this physical interpretation of complex roots is of general application.* A consideration of the analogous problem in three dimensions suggests, however, that an attempt to base a proof on geometrical considerations may fail.

3. *Study of a particular equation.*

When $\sigma_s = \tau_s = 0$ and the quantities λ_s, μ_s are all positive integers the transcendental equation is of a type which occurs in the solution of problems of vibration and in problems of the conduction of heat.

Let us write the equation in the form

$$(x^r + p_1x^{r-1} + p_2x^{r-2} + \cdots p_r)\cos\nu\sin(x-\theta) + (x^r + q_1x^{r-1} + q_2x^{r-2} + \cdots q_r)\sin\nu\cos(x-\theta) = 0$$

where ν, p_s, q_s, are real constants. This equation is of the form (3) if

$$\cos\nu\,(x^r + p_1x^{r-1} + \cdots p_r) + i\sin\nu\,(x^r + q_1x^{r-1} + \cdots q_r) \equiv f(x) + ig(x)$$

where $f(x)$ and $g(x)$ are polynomials such that the algebraic equation

$$f(x) + ig(x) = 0$$

has all its roots of the form $a_s + ib_s$ where $b_s > 0$.

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*A stream line may end at a sink but by supposition the sinks all lie on or below the axis of x consequently when there are no vortices a stream line must be expected to meet the x-axis.

Writing $x = -iy$ the corresponding equation in y must have only pseudo-negative roots, a pseudo-negative root being defined to be one whose real part is negative.*

Considering first the case in which $r = 1$ the equation for y is

$$e^{iy}y + i(p_1 \cos\nu + iq_1 \sin\nu) = 0$$

and the condition for pseudo-negative roots is $\sin\nu\cos\nu\,(p_1 - q_1) > 0$. When this condition is satisfied the roots of the equation

$$(x + p_1)\cos\nu\sin(x-\theta) + (x+q_1)\sin\nu\cos(x-\theta) = 0$$

are all real and increase with θ.

Next, consider the case $r = 2$. The equation for y is

$$\cos\nu\,(-y^2 - ip_1y + p_2) + i\sin\nu\,(-y^2 - iq_1y + q_2) = 0.$$

Writing this in the form

$$y^2 + (P_1 + iP_2)y + Q_1 + iQ_2 = 0$$

where P_1, P_2, Q_1, Q_2 are all real the conditions for pseudo-negative roots are †

$$P_1 > 0, \quad P_1^2Q_1 + P_1P_2Q_2 - Q_2^2 > 0.$$

Hence, when

$$(p_1 - q_1)\sin\nu\cos\nu > 0 \quad (p_1 - q_1)(p_2q_1 - p_1q_2) > (p_2 - q_2)^2$$

the roots of the equation ‡

$$(x^2 + p_1x + p_2)\cos\nu\sin(x-\theta) + (x^2 + q_1x + q_2)\sin\nu\cos(x-\theta) = 0$$

are all real and increase with θ.

When the coefficients of an algebraic equation

$$H(y) \equiv A_0y^r + A_1y^{r-1} + A_2y^{r-2} + \cdots A_r = 0$$

are all real the necessary and sufficient conditions for pseudo-negative roots

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* Chipart and Liénard, *Comptes Rendus*, t. 157, pp. 691, 838.

† See for instance H. Bateman, *Stability of the Parachute and Helicopter*, N. A. C. A. Report, No. 80 (1920).

‡ If β_1 and β_2 are positive the roots of the equation

$$(\beta_1 + \beta_2)x\cos(x-\theta) + (\beta_1\beta_2 - x^2)\sin(x-\theta) = 0$$

are all real and increase with θ. The case in which $\theta = 0$ is discussed by Koshliakov.

have been investigated by E. J. Routh * and other writers.† The conditions are required in investigations concerning the stability of airplanes and have been expressed in the following form by Hurwitz

$$A_1/A_0 > 0, \quad \begin{vmatrix} A_1 & A_3 \\ A_0 & A_2 \end{vmatrix} \; 1/A_0 > 0 \qquad \begin{vmatrix} A_1 & A_3 & A_5 \\ A_0 & A_2 & A_4 \\ 0 & A_1 & A_3 \end{vmatrix} \; 1/A_0 > 0, \cdots$$

The rule for forming these determinants is that a suffix increases by two units for each step to the right and decreases by one unit for each step in a downward direction. When s is negative or greater than r, $A_s = 0$.

When the coefficients of the equation are complex the conditions for pseudo-negative roots are not so simple. They may, however, be found step by step with the aid of the following theorem due to Schur.† Let

$$\bar{H}^*(y) = \bar{A}_r - \bar{A}_{r-1}y + \bar{A}_{r-2}y^2 + \cdots (-1)^r \bar{A}_0 y^r$$

where $\bar{A}_s$ is the complex quantity conjugate to A_s then, when the equation $H(y) = 0$ has only the pseudo-negative roots $-x_s + iy_s$, we have

$$H^*(z) = (-1)^r \bar{A}_0(-z + x_1 + iy_1)(-z + x_2 + iy_2) \cdots$$
$$H(z) = A_0(z + x_1 - iy_1)(z + x_2 - iy_2) \cdots$$

and so if $z = x + iy$

$$|H(z)| = |A_0|\,[(x + x_1)^2 + (y - y_1)^2]^{\frac{1}{2}}\,[(x + x_2)^2 + (y - y_2)^2]^{\frac{1}{2}} \cdots,$$
$$|H^*(z)| = |A_0|\,[(x - x_1)^2 + (y - y_1)^2]^{\frac{1}{2}}\,[(x - x_2)^2 + (y - y_2)^2]^{\frac{1}{2}} \cdots :$$

consequently $|H(z)|$ is greater, equal to or less than $|H^*(z)|$ according as x is positive, zero or negative.

Now let $|c| > |\gamma|$ where c and γ are complex constants, then the equation $H(z) = 0$ has pseudo-negative roots when and only when the equation

$$I(z) \equiv cH(z) - \gamma H^*(z) = 0$$

has pseudo-negative roots.

If in fact $I(z) = 0$ when z is pseudo-positive we find that

$$|H^*(z)| > |H(z)|$$

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* *Rigid Dynamics*, Vol. II, p. 194; *Stability of Motion*, London, 1877, p. 74.

† A. Hurwitz, *Math. Ann.*, Bd. 46 (1895), p. 273; L. Orlando, *Ibid.*, Bd. 71 (1912), p. 233; Chipart and Liénard, *loc. cit.*; M. Fujiwhara, *The Tohoku Math. Journ.* (1915), Vol. 7, p. 78; J. Schur, *Zeitschrift für angew. Math. und Mech.*, Bd. 1 (1921), p. 307.

which is contrary to the previous statement. To prove that $H(z) = 0$ has pseudo-negative roots when $I(z) = 0$ has roots of this character we remark that the relation between the two functions $H(z)$, $I(z)$ is a reciprocal one. In fact if $\bar{c}$, $\bar{\gamma}$, are the complex quantities conjugate to c and γ respectively we have

$$I^*(z) = \bar{c}\,H^*(z) - \bar{\gamma}\,H(z)$$
$$H(z) = c_1 I(z) - \gamma_1 I^*(z)$$

where

$$c_1 = \frac{\bar{c}}{|c|^2 - |\gamma|^2}, \qquad \gamma_1 = -\frac{\gamma}{|c|^2 - |\gamma|^2}$$

$$|c_1|^2 - |\gamma^2| = \frac{1}{|c|^2 - |\gamma|^2} > 0$$

It should be noticed that the reciprocal relation breaks down when $|c|^2 = |\gamma|^2$ and then we can only infer by the above reasoning that $I(z) = 0$ has pseudo-negative roots when the roots of $H(z) = 0$ are pseudo-negative.

If ξ is a pseudo-negative quantity such that $|H^*(\xi)| > |H(\xi)|$ we may take $c = H^*(\xi)$, $\gamma = H(\xi)$ and then the equation

$$I(z) = H(z)\,H^*(\xi) - H^*(z)H(\xi) = 0$$

has the root $z = \xi$ and so the equation $H(z) = 0$ will have pseudo-negative roots when and only when the equation of lower degree

$$\frac{H(z)\,H^*(\xi) - H^*(z)\,H(\xi)}{z - \xi}$$

has only pseudo-negative roots.

Schur's theorem may be used in connection with Koshliakov's theorem as follows:

Let $$H(y) = f(-iy) + ig(-iy)$$

where

$$f(x) + ig(x) = \prod_{s=1}^{r}(x - z_s)$$

is a particular case of equation (2), then

$$H^*(y) = f(-iy) - ig(-iy)$$

and if $c^2 \geq \gamma^2$ Schur's theorem tells us that the equation

$$(c - \gamma)f(x) + i(c + \gamma)g(x) = 0$$

has roots of type $u+iv$ where $v>0$ and that the roots of the equation

$$(c+\gamma)g(x)\cos(x-\theta)+(c-\gamma)f(x)\sin(x-\theta)=0$$

are all real and increase with θ. In particular, when $c=\pm\gamma$ the theorem tells us that the roots of the equations $f(x)=0$, $g(x)=0$ are all real. This theorem looks as if it should be true in the more general case in which

$$f(x)+ig(x)=\prod_{s=1}^{n}\frac{(x-a_s-ib_s)^{\lambda_s}}{(x-a_s+i\beta_s)^{\mu_s}}$$

and

$$b_s>0, \quad \beta_s>0, \quad \lambda_s>0, \quad \mu_s>0.$$

4. *The effect of a conformal transformation.*

The theorem just mentioned appears as a particular case of the general theorem of § 1 when we change the scale by means of a conformal transformation of type $z=CZ$ so that ψ_1 becomes

$$\psi_1=V(x\cos\omega-y\sin\omega)$$

and then make $V\to 0$.

A limiting form of the theorem is obtained also by making a conformal transformation of type

$$z=\frac{AZ+B}{CZ+D}, \qquad z_s=\frac{AZ_s+B}{CZ_s+D}, \qquad AD-BC>0$$

with real coefficients A, B, C, D. Since

$$z-z_s=\frac{(AD-BC)(Z-Z_s)}{(CZ+D)(CZ_s+D)}$$

we see that if $\sum m_s=\sum k_s$

$$\phi_0+i\psi_0=\sum_{s=1}^{n}m_s\log(z-z_s)-\sum_{s=1}^{n}k_s\log(z-z_s).$$

The rectilinear flow specified by $\phi_1+i\psi_1=-z\sec\omega\, e^{-i\omega}$ is transformed into the flow due to a double source on the axis of x the source being above the axis and the sink below. When $\omega\neq 0$ there is a double vortex associated with the double source but when $\omega=0$ there is not.

Let us next consider the effect of the conformal transformation

$$2z=Z+h^2/z, \qquad 2z_s=Z_s+h^2/z_s,$$

which is used so advantageously in the theory and derivation of Joukowsky aerofoils. This transformation sends a point above the axis of x into a point above the axis of y when a certain inequality is satisfied. In fact if $z = x + iy$, $Z = X + iY$ we have

$$2y = Y\left[1 - \frac{h^2}{X^2 + Y^2}\right]$$

Hence if $X^2 + Y^2 > h^2$ we have $y > 0$ when $Y > 0$ but if $X^2 + Y^2 < h^2$ we have $y > 0$ when $Y < 0$.

We also have the relation

$$2(z - z_s) = (Z - Z_s)(1 - h/ZZ_s)$$

Writing

$$Z' = h^2/Z, \quad Z'_s = h^2/Z_s, \quad 2z = Z' + h^2/Z'$$

we find that

$$2(z - z_s) = \frac{(Z - Z_s)(Z - Z'_s)}{Z}$$

$$\sum_{s=1}^{n} m_s \log(z - z_s) = \sum_{s=1}^{n} m_s[\log(Z - Z_s) + \log(Z - Z'_s) - \log 2Z].$$

If Z_s lies above the axis of x and outside the circle $X^2 + Y^2 = h^2$. Z'_s lies below the axis of x and inside the circle $X^2 + Y^2 = h^2$. We thus have the following theorem:

If

$$\prod_{s=1}^{n} [(x - Z_s)(x - Z'_s)]^{\lambda_s} = f(x) + ig(x)$$

where x is real and $\lambda_s > 0$ then the roots of the equations $f(x) = 0$, $g(x) = 0$ are all real. In particular, this result shows that the reality of the roots of $f(x)$ and $g(x)$ is not sufficient to ensure that all the roots of $f(x) + ig(x) = 0$ will be of the form $a_s + ib_s$ where $b_s > 0$.

The relation

$$\frac{z - z_s}{z - z_t} = \frac{(Z - Z_s)(Z - Z'_s)}{(Z - Z_t)(Z - Z'_t)}$$

shows that the stream lines for a source and sink are transformed into the stream lines of two equal sources and two equal sinks. The two sources are, however, connected by the relation $Z_s Z_s' = h^2$ and the sinks by a similar relation $Z_t Z_t' = h^2$. Denoting the sources by P, P' respectively, the points $\pm h$ by H, H' respectively and the sinks by Q, Q' respectively we note that P, P',

H, H' form a harmonic set of concyclic points and that likewise Q, Q', H, H', form a harmonic set of concyclic points. Let us now suppose that P, P', Q, Q' are given and that H and H' are to be found. Using letters p, p', q, q', h, h' to denote their complex coordinates relative to some set of axes the conditions for harmonic sets of points give

$$[h-\tfrac{1}{2}(p+p')]\,[h'-\tfrac{1}{2}(p+p')]=\tfrac{1}{4}(p-p')^2$$
$$[h-\tfrac{1}{2}(q+q')]\,[h'-\tfrac{1}{2}(q+q')]=\tfrac{1}{4}(q-q')^2.$$

These equations give hh' and $h+h'$ uniquely and so H and H' are uniquely determined. The stream lines for two equal sources and two equal sinks may thus be determined from the stream line for a source and sink by means of the conformal transformation.*

5. *Motion in planes through the axis of x.*

It seems worth while to study the types of equations which arise when stream lines are studied with the aid of Stokes' current function ψ. Taking first the case in which there are a number of sources on the axis of x to the left of the origin and a number of sinks on the same axis but to the right of the origin the stream function is †

$$\psi=+\sum_{s=1}^{n}\Big[\lambda_s\left(\frac{x_s+x}{\sqrt{(x+x_s)^2+y^2+z^2}}-1\right)+\mu_s\left(1-\frac{x-x_s}{\sqrt{(x-x_s)^2+y^2+z^2}}\right)\Big]$$

Adding to this the stream function $\psi=-\tfrac{1}{2}U(y^2+z^2)$ for a steady rectilinear flow parallel to the axis of x and writing $y^2+z^2=\omega^2$ we find that the points in which the stream lines $\psi=-\theta$ meet the plane $x=0$ are given by

$$\tfrac{1}{2}U\omega^2-\sum_{s=1}^{n}(\lambda_s+\mu_s)\frac{x_s}{\sqrt{x_s^2+\omega^2}}=\theta.$$

In general the roots of this equation in ω are not all real but there is at least one real root between $\sqrt{2\theta/U}$ and ∞. To find how this root varies with θ we differentiate and find

* For a discussion of the form of the stream-lines reference may be made to a paper by Rankine, *Phil. Trans.* (1871), p. 267.

† The quantities λ_s, μ_s, x_s, U are supposed to be all positive. For each value of s either λ_s or μ_s may be taken to be zero unless a sink happens to be the image of a source in the plane $x=0$.

$$\frac{d\theta}{d\omega} = U\omega + \sum_{s=1}^{n} (\lambda_s + \mu_s) \frac{\omega x_s}{(x_s^2 + \omega^2)^{3/2}}$$

A similar result may be obtained when sources and sinks are distributed over circular areas having the axis of s as plane of symmetry. For illustration we shall consider simply the case in which sources are distributed over the circle $\omega^2 < a^2$, $x = 0$ with a density proportional to $(a^2 - \omega^2)^{-\frac{1}{2}}$.

Writing $$x = a\mu\zeta, \qquad \omega = a(1-\mu^2)^{\frac{1}{2}}(1+\zeta^2)^{\frac{1}{2}}$$

and assuming that there is also a superposed steady stream parallel to the axis of x, an appropriate stream function is

$$\psi = -\tfrac{1}{2}U\omega^2 + Aa\mu.$$

The points in which the stream lines $\psi = -\theta$ cut the plane $x = c$ are given by

$$\tfrac{1}{2}Ua^2(1-\mu^2)\left(1 + \frac{c^2}{a^2\mu^2}\right) - Aa\mu = \theta$$

and as θ varies we have

$$\frac{d\theta}{d\mu} = -Aa - Ua^2\mu\left(1 + \frac{c^2}{a^2\mu}\right) - \frac{Uc^2}{\mu^3}(1-\mu^2),$$

and $d\omega/d\mu$ is negative, consequently ω increases with θ, provided Aa and c have the same sign, ζ being positive so that μ has the same sign as c.

6. *Properties of the Jacobian.*

When the density remains constant the equations of motion of a viscous fluid

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial V}{\partial x} + \nu\nabla^2 u$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial V}{\partial y} + \nu\nabla^2 v$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

give

$$\nabla^2\Omega = 2J$$

where

$$\Omega = V + p/\rho, \qquad J = \frac{\partial(u,v)}{\partial(x,y)}$$

Integrating round a closed curve we have also the relation

$$\int\left(\frac{\partial\Omega}{\partial x}\,dy - \frac{\partial\Omega}{\partial y}\,dx\right) = \int(u\,dv - v\,du).$$

The quantity J can also be expressed in the form

$$J = -\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial u}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

and this relation shows that when the quantity $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ is zero J is either zero or negative. If ζ is everywhere finite and continuous it is clear that it has one sign throughout a simply connected region within which J is positive because at a point where J vanishes and changes sign J is either negative or zero. There may, of course, be more than one region within which J is positive and it does not follow that the sign of J is the same for different regions of this type.

The sign of J may be studied with the aid of the surface whose equation in rectangular coordinates is

$$z = \psi(x, y)$$

ψ being the stream function defined by the equation

$$d\psi = v\,dx - u\,dy$$

when t is constant. Since

$$-J = \frac{\partial^2\psi}{\partial x^2}\,\frac{\partial^2\psi}{\partial y^2} - \left(\frac{\partial^2\psi}{\partial x \partial y}\right)^2$$

we find that in orthogonal projection on the plane of x and y the synclastic portion of the surface projects into the region or regions for which $J > 0$, the anticlastic portion of the surface projects into the region or regions for which $J < 0$ and the parabolic curve projects into the curve $J = 0$.

A stationary boundary of a moving viscous fluid is generally a part of the curve $J = 0$ for along this boundary $u = 0$, $v = 0$ and so

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0, \qquad \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy = 0,$$
$$J = 0.$$

If a rigid boundary has a simple velocity of translation the preceding equations still hold but if it has a rotational motion we can no longer say this. Indeed, if the axis of rotation passes through the origin the boundary condition is of type.

$$u = -\omega x, \qquad v = \omega y$$

and we have

$$\left(\frac{\partial u}{\partial x} + \omega\right) dx + \frac{\partial u}{\partial y}\, dy = 0$$

$$\frac{\partial v}{\partial x}\, dx + \left(\frac{\partial v}{\partial y} - \omega\right) dy = 0$$

$$J = \omega\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \omega^2.$$

In steady motion the places at which a particle of fluid has no acceleration generally lie on the curve $J = 0$, for we then have

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = 0. \qquad u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = 0,$$

An exception to this rule may arise when $u = v = 0$ at the place in question.

At a point on the curve $J = 0$ we can generally write

$$\frac{\dfrac{\partial u}{\partial x}}{lm} = \frac{\dfrac{\partial u}{\partial y}}{-l^2} = \frac{\dfrac{\partial v}{\partial x}}{m^2} = \frac{\dfrac{\partial v}{\partial y}}{-lm} = \zeta$$

where (l, m) are the direction cosines of the common tangent to the two curves $u =$ constant, $v =$ constant, which pass through the point. That these curves do have a common tangent is seen from the equations

$$l\,\frac{\partial u}{\partial x} + m\,\frac{\partial u}{\partial y} = 0, \qquad l\,\frac{\partial v}{\partial x} + m\,\frac{\partial v}{\partial y} = 0.$$

Thus the parabolic curve $J = 0$ is partly or wholly a locus of points at which a curve $u =$ constant touches a curve $v =$ constant. As a point moves away from the parabolic curve

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial v}{\partial x} = m\zeta(lu + mv)$$

and so if $u^2 + v^2 = q^2$ we have

$$\frac{\partial q}{\partial x} = m\zeta\cos\alpha = \zeta\cos\alpha\sin\beta$$

where α is the angle between the direction of the velocity q and the line (l, m) while β is the angle which this last line makes with the axis of x. Since, moreover, the axis of x may be taken in any direction we have an expression for the rate of variation of q in any direction. The rate of variation is zero along the line (l, m) and a maximum in a direction perpendicular to the line.

As a point moves along the curve $J=0$ it may come to a place where either $\cos\alpha=0$ or $\zeta=0$ and then all the first derivatives of q vanish. At such a point q is stationary in value. There may also be points on the curve $J=0$ at which v/u is stationary in value.

Considering the case of steady motion we have

$$2yJ=\frac{\partial}{\partial x}\left[y\left(u\frac{\partial v}{\partial y}-v\frac{\partial u}{\partial y}\right)\right]-\frac{\partial}{\partial y}\left[y\left(u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}\right)\right]+u\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial x}.$$

Integrating over the region bounded by a closed curve C and a circle of infinite radius we can transform the integral of the right hand side into a line integral by replacing $\partial u/\partial x$ by $-\partial v/\partial y$ and then writing

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=\nu\nabla^2 v-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{\partial V}{\partial y}$$

If l, m denote the direction cosines of the outward normal to a curve, ds the element of arc

$$\begin{aligned}-2\rho\int\int yJ\,dxdy&=\int_C\{\rho y(udv-vdu)+\tfrac{1}{2}\rho q^2 m\,ds\\&\quad+ds[\mu(l\frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y})-m(p+\rho V+\tfrac{1}{2}\rho q^2)]\}\\&\quad-\int_\infty[\rho y(udv-vdu)+\tfrac{1}{2}\rho q^2 m\,ds]\\&\quad-\int_\infty ds\,[\mu(l\frac{\partial v}{\partial x}+m\frac{\partial v}{\partial y})-m(p+\rho V+\tfrac{1}{2}\rho q^2)].\end{aligned}$$

The last integral may be omitted if $p+\rho V+\frac{1}{2}\rho q^2$ can be regarded as constant at infinity and $\partial v/\partial x$, $\partial v/\partial y$ of order $1/r^2$. We shall suppose in fact that

$$u\sim U-\frac{\kappa y}{r^2},\qquad v\sim V+\frac{\kappa x}{r^2}$$

then

$$\begin{aligned}&\int_\infty[\rho y(udv-vdu)+\tfrac{1}{2}\rho q^2 m\,ds]\\&=U\rho\int_\infty(ydv-\kappa\frac{myds}{r^2})+V\rho\int(\kappa\frac{mxds}{r^2}-ydu)\\&=V\rho\int(udy+\kappa\frac{mxds}{r^2})-U\rho\int(vdy+\kappa\frac{myds}{r^2})\\&=-\kappa V\rho\int\frac{xdx+ydy}{r^2}-\kappa U\rho\int\frac{xdy-ydx}{r^2}=-2\pi\kappa U\rho.\end{aligned}$$

When $\mu = 0$ and $p + \rho V + \frac{1}{2}\rho q^2$ is constant we have simply

$$-2\rho \iint yJ\,dxdy = 2\pi\kappa U\rho + \int_C [\rho y(udv - vdu) + \tfrac{1}{2}\rho q^2 m ds].$$

If, moreover C is a circle of radius a and is at the same time a stream line we may write $u = q\cos\theta$, $v = q\sin\theta$

$$udv - vdu = q^2 d\theta = +\, q^2 \frac{ds}{a}$$

$$\int_C [\rho y(udv - vdu) + \tfrac{1}{2}\rho q^2 m ds] = +\, \tfrac{3}{2}\int \rho q^2 m ds$$

$$= 3Y$$

where Y is the force component in the direction of the y-axis which the fluid exerts on unit length of the cylinder with C as cross section. Consequently

$$-2\rho \iint yJ\,dxdy = 2\pi\kappa U\rho + 3Y.$$

If the cylinder is the only obstacle $Y = -2\pi\kappa U\rho$ and so

$$\rho \iint yJ\,dxdy = 2\pi\kappa U\rho = -Y.$$

Similarly, if X is the x-component of the force

$$\rho \iint xJ\,dxdy = -2\pi\kappa V\rho = -X.$$

To verify these results in the case when $V = 0$ we note that

$$-\psi = Uy\left(1 - \frac{a^2}{r^2}\right) - \kappa \log \frac{r}{a}$$

$$J = -\left[\frac{k^2}{r^4} - \frac{4a^2 U\kappa y}{r^6} + \frac{4a^4 U^2}{r^6}\right]$$

$$\iint xJ\,dxdy = 0, \qquad \iint yJ\,dxdy = 2\pi\kappa U.$$

In the case of a viscous fluid we have the relation

$$\iint 2J\,dxdy = \int_\infty (udv - vdu) - \int_C (udv - vdu)$$

If C is a stationary obstacle over which u and v both vanish and if at infinity $u \to U$, $v \to V$, the differences $u - U$, $v - V$ being of order $1/r$, both line integrals vanish and so $\iint J\,dxdy = 0$. This equation tells us that J cannot have the same sign at all points outside C. Now on C we have $J = 0$, consequently J must become positive within some domain surrounding C and will be zero on another curve which either surrounds C or meets it. For illus-

tration let us consider Lamb's approximate solution of the problem of viscous flow round a circular cylinder * of radius a. At points near the cylinder we have

$$u = \tfrac{1}{2}C\{\gamma - \tfrac{1}{2} + \log(\tfrac{1}{2}kr) + \tfrac{1}{2}(r^2 - a^2)\frac{\partial^2}{\partial x^2}(\log r)\}$$

$$v = \tfrac{1}{4}C(r^2 - a^2)\frac{\partial^2}{\partial x \partial y}\log r, \qquad 2U = C[\tfrac{1}{2} - \gamma - \log(\tfrac{1}{2}ka)]$$

$$J = -\frac{c^2}{4r^6}(r^2 - a^2)(x^2 - y^2 - a^2).$$

Hence near the cylinder J is zero on the cylinder and on the rectangular hyperbola $x^2 - y^2 = a^2$.

A point of inflexion on a stream line must lie within the region for which J is negative, for its coordinates satisfy the equation

$$\left(\frac{\partial\psi}{\partial y}\right)^2 \frac{\partial^2\psi}{\partial x^2} - 2\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x \partial y} + \left(\frac{\partial\psi}{\partial x}\right)^2 \frac{\partial^2\psi}{\partial y^2} = 0$$

or

$$\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right)^2 = v^2\left[\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right] = -v^2 J.$$

When J is positive the vorticity must have an appreciable value, for we have the inequality

$$\zeta^2 - 4J = \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)^2 + 4\left(\frac{\partial u}{\partial x}\right)^2 > 0.$$

This result may be illustrated by the case of the motion of a viscous fluid between two coaxal circular cylinders. Writing

$$u = -\omega y, \qquad v = \omega x$$

where †

$$\omega = \frac{A}{r^2} + B$$

we find that

$$J = (B + \frac{A}{r^2})(B - \frac{A}{r^2}), \qquad \zeta = 2B.$$

If $\omega = \omega_0$ when $r = a$ and $\omega = \omega_1$ when $r = b$ we have

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* *Hydrodynamics*, Ch. XI, p. 582.

† See Lamb's *Hydrodynamics*, Ch. XI, p. 556.

$$A = \frac{a^2b^2(\omega_0 - \omega_1)}{b^2 - a^2}, \qquad B = \frac{b^2\omega_1 - a^2\omega_0}{b^2 - a^2}$$

$$\left(B + \frac{A}{a^2}\right)\left(B - \frac{A}{a^2}\right) = \frac{2b^2\omega_0\omega_1 - (a^2 + b^2)\omega_0^2}{(b^2 - a^2)^2}$$

$$\left(B + \frac{b^2}{A}\right)\left(B - \frac{A}{b^2}\right) = \frac{(a^2 + b^2)\omega_1^2 - 2a^2\omega_0\omega_1}{(b^2 - a^2)^2}$$

If $b > a$ and $\omega_1 > \omega_0$, ζ (or $2B$) is large enough to make J positive. If $\omega_1 = 0$ J is negative except on $r = b$.

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